1. **[6]** Quincy writes the numbers 1, 3, 4, 8, and 9 on a chalkboard. Every minute, he replaces two numbers m and n on the chalkboard with $5m+n$. Compute the maximum possible value of the final number on the chalkboard.

Team Name: ____________________ Answer: ____________________

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2. **[6]** Compute the number of positive integers $n < 50$ such that both n and $50 - n$ have an even number of divisors.

Team Name: ____________________ Answer: ____________________

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3. **[7]** Percy rolls eight fair six-sided dice and records their values a, b, c, d, e, f, g, and h. If P is the probability that

$$(a+b)^{c+d}(e+f)^{g+h} = 2025,$$

and $P = p^a q^b r^c$ where p, q, r are distinct primes and a, b, c are nonzero integers, compute $pa + qb + rc$.

Team Name: ____________________ Answer: ____________________

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4. **[7]** Let $ABCDE$ be a pentagon and suppose $AB \parallel CD$ and $BC \parallel DE$. Compute the area of this pentagon given that $AB = BC = 15$, $CD = 21$, $DE = 5$, and $EA = 8$.

Team Name: ____________________ Answer: ____________________

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5. **[8]** Let $x = 1252^3 - 1248^3$. Compute $\sqrt{x - 16}$.

Team Name: ____________________ Answer: ____________________

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6. **[8]** Compute

$$\sum_{m|1024} \sum_{n|1024} \left\lfloor \frac{m}{n} \right\rfloor.$$

Team Name: ____________________ Answer: ____________________

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7. **[9]** Let ABC be a triangle with $AB = 1$, $BC = \sqrt{3}$, and $CA = 2$. Let O be the circumcircle of ABC and let ℓ be the tangent to O parallel to AC and closer to B. Suppose ℓ intersects O at P. Compute the area of triangle OPB.

Team Name: ____________________ Answer: ____________________

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8. **[9]** Call a set $S \subset \{1, 2, \ldots, 2025\}$ *corwizzy* if for any $a, b \in S$ with $ab \leq 2025$, $ab \in S$. Compute the smallest positive integer $n > 1$ such that there exists a corwizzy set S whose elements sum to n.

Team Name: ____________________ Answer: ____________________

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9. **[10]** Let O be a circle and $XY = 8$ be a diameter of O. Let A lie on O and let the circle with center X and radius XA intersect O at $B \neq A$. Compute the maximum possible area of triangle ABY.

Team Name: ____________________ Answer: ____________________

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10. **[Up to 10]** Submit an ordered quadruple of integers (a, b, c, d) with $0 \leq a, b, c, d \leq 50$. Let $\mathcal{R}$ be the axis-aligned rectangle who's bottom left vertex has coordinates (a, b) and who's upper right vertex has coordinates (c, d). If $\mathcal{R}$ completely coincides with another teams rectangle (i.e. is exactly the same rectangle) or if $\mathcal{R}$ has intersection of non-zero area with another team's rectangle and neither contain each other, you get 0 points. Otherwise, you get $\frac{|b-a||d-c|}{250}$ points.

Team Name: ____________________ Answer: ____________________

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11. **[11]** Given a circular mirror, at how many angles strictly above the horizontal from the south pole can a laser be shot such that it bounces at most 20 times before returning?

Team Name: ____________________ Answer: ____________________

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12. **[11]** What is the largest positive integer k such that there exist k consecutive four-digit positive integers, each with no more than 3 distinct digits?

Team Name: ____________________ Answer: ____________________

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13. **[12]** An 8×9 grid is filled with the 72 divisors of 36000, randomly and without replacement. CrussoCode randomly and uniformly picks one of the squares, and computes its prime factorization along with the prime factorizations of its neighbors (only lateral neighbors, not diagonal). What is the probability that each such prime factorization has pairwise distinct exponents for each prime factor?

(For example, $2^3 3^3$ and $2^4 3^2$ have pairwise distinct exponents as $3 \neq 4$ and $3 \neq 2$, but $2^4 3^2$ and $2^3 3^2$ do not, as $2 = 2$.)

Team Name: ____________________ Answer: ____________________

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14. **[12]** For how many ordered triples of integers (a, b, c) satisfying $0 \leq a, b, c \leq 20$ do there exist integers $(x, y, z) \neq (0, 0, 0)$ satisfying

$$\begin{aligned} ax &= by + cz \\ bx &= cy + az \\ cx &= ay + bz? \end{aligned}$$

Team Name: ____________________ Answer: ____________________

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15. **[13]** Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let X be the foot of A onto BC and let D and E be the feet of X onto AB and CA respectively. Compute DE.

Team Name: ____________________ Answer: ____________________

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16. **[13]** Let a_n, b_n be sequences with $a_1 = 6$, $b_1 = 7$ and

$$\begin{aligned} a_{n+1} &= {b_n}^2 + a_n b_n \\ b_{n+1} &= {a_n}^2 + a_n b_n \end{aligned}$$

for all $n \geq 1$. Compute the number of divisors of a_{10}.

Team Name: ____________________ Answer: ____________________

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17. **[14]** Mario starts at $(0, 0, 0)$ and is trying to reach Peach's CASTLE at $(41, 6, 7)$. Every time he moves, Mario can do one of the following:

    (i) Move $+2$ units in the x direction

    (ii) Choose two of the x, y, z-directions and move $+1$ units in both of them

    Let N be the number of distinct paths that Mario can take to Peach's CASTLE. Find the sum of the distinct prime factors of N.

Team Name: ____________________ Answer: ____________________

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18. **[14]** Let f be a function such that $f(p^k) = p^k + p^{k-1}$ for prime p and positive integer k and $f(ab) = f(a)f(b)$ for all relatively prime integers a, b. Find the sum of all possible distinct values of $\frac{f(2024x)}{2024f(x)}$ for integer x.

Team Name: ____________________ Answer: ____________________

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19. **[15]** Let triangle ABC have $AB = 5$, $BC = 6$, and $CA = 7$ and P be a point in the plane. If X, Y, Z are the feet of the altitudes from P to AB, BC, and CA, find the minimum possible value of $PX^2 + PY^2 + PZ^2$ across all P.

Team Name: ____________________ Answer: ____________________

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20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put Y if you think the statement is true, N if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.

(a) Let ABC be a triangle and A' be the reflection of A over BC. Define B' and C' analogously. It is possible for triangle ABC to be strictly inside of triangle $A'B'C'$.

(b) Given triangle ABC, there always exists a point P in the same plane such that $PA : PB : PC = 41 : 67 : 69$.

(c) Put an infinite, axis-aligned grid on the coordinate plane. Let a *phone* be a contiguous path of squares of the grid (that is, no 2×2 set of squares is completely on the path). Call a phone *new* if it is possible to get from one end of the phone to the other while staying on the squares of the phone and only moving up or to the right. Any new phone can tile the grid.

(d) Let $f(x) = \frac{ax+b}{cx+d}$ and $g(x) = \frac{px+q}{rx+s}$ be functions with real coefficients on the reals such that $f(f(x))$ and $g(g(x))$ are not the identity. Suppose $f(g(x)) = g(f(x))$ for all x. Then $f(x) - x$ and $g(x) - x$ have the same roots.

(e) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \sin^6\left(x - \frac{3\pi}{8}\right) + \sin^6\left(x - \frac{\pi}{8}\right) + \sin^6\left(x + \frac{\pi}{8}\right) + \sin^6\left(x + \frac{3\pi}{8}\right).$$

For any $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that $0 < |x - y| < \frac{\pi}{8}$ and $f(x) = f(y)$.

(f) For all positive integers $n > 1$ and $k < n$, $\gcd\left(\binom{n}{k}, n\right) > 1$.

Team Name: ____________________ Answer: ____________________

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21. **[16]** For positive integers n and k, let $\psi_k(n)$ be the smallest positive integer such that $n\psi_k(n)$ is a perfect kth power. Call a positive integer n *psichotic* if there exists a square-free positive integer N such that

$$n = \sum_{d \mid N^{104}} \varphi(\psi_3(d))\varphi(\psi_5(d))\varphi(\psi_7(d)).$$

Find the number of positive integers $n \le 2025^6$ that are psichotic (recall that for positive integers n, $\varphi(n)$ is the number of integers relatively prime to n between 1 and n inclusive).

Team Name: ____________________ Answer: ____________________

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22. **[16]** Compute the maximum value of

$$f(x) = \frac{x^2 + x - 2}{x^4 + 2x^3 - x^2 - 2x + 3}$$

as x ranges over $\mathbb{R}$.

Team Name: ____________________ Answer: ____________________

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23. **[17]** Aditya tags each point (a, b) where a and b are integers with the number $a^2 + b^2$. Andrew starts at $(0, 0)$ and every minute, he walks either up 1 unit or right 1 unit with equal probability. After 22 minutes, Megan sums the tagged numbers on each point of Andrew's walk. What is the expected value of this sum?

Team Name: ____________________ Answer: ____________________

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24. **[17]** How many ordered pairs of positive integers (a, b) with $a > b$ and $a + b < 1000$ have an integer solution x to

$$\sqrt{a - x} + (b + x) = (a - x) - \sqrt{b + x}?$$

Team Name: ____________________ Answer: ____________________

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25. **[18]** Compute all ordered pairs of integers (a, b) such that

$$\binom{289}{a + b} = 9^8(b^2 - 9) - 2b + 7.$$

Team Name: ____________________ Answer: ____________________

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26. **[18]** Let ABC be a triangle with $AB = 7$, $AC = 8$, and $BC = 5$. Let P be an arbitrary point on $\overline{BC}$ and E and F be on $\overline{AB}$ and $\overline{AC}$, respectively, such that $BE = BP$ and $CF = CP$. If D is the reflection of P over the angle bisector of $\angle A$, find the minimum possible area of $\triangle DEF$.

Team Name: ____________________ Answer: ____________________

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27. **[19]** Consider the 60-edge graph C consisting of a 40-cycle and edges between antipodal vertices. Let A and B be neighboring vertices (not antipodes) and let A' and B' be their respective antipodes. Let a *borisaurus* be a partition of a graph's vertex set into connected 4-element subsets (i.e. the maximal subgraph containing the four vertices is connected). How many borisauruses on C satisfy the additional condition that if both A and B or both A' and B' are in the same subset S of the borisaurus, then that set is precisely $\{A, B, A', B'\}$?

Team Name: ____________________ Answer: ____________________

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28. **[19]** Let the multivariable polynomial $P_n(x, y)$ be defined by

$$P_n(x,y) = \sum_{i=0}^{n} x^i y^{n-i}$$

for all n. Compute the number of ordered pairs of complex numbers (x, y) for which $P_{14}(x+1, y+4) = P_{23}(x+2, y+3) = 0$.

Team Name: ____________________ Answer: ____________________

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29. **[20]** Let $ABCD$ be a cyclic quadrilateral with AD and BC meeting at T, AC and BD meeting at P, and AB and CD meeting at a $60°$ angle. Point $M \neq P$ is the intersection of the circumcircles of $\triangle PAB$ and $\triangle PCD$ and V is the reflection of M over CD. Given that $\frac{AB}{CD} = \frac{3}{5}$ and $MV = 192$, compute TV.

Team Name: ____________________ Answer: ____________________

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30. **[20]** For primes p, let $f(p)$ be the number of ordered pairs (a, b) of integers such that $0 \leq a, b < p$ and p divides $a^5 - 10a^3b^2 + 10a^2b^3 - b^5 - 1$ and $5a^4b - 10a^3b^2 + 5ab^4 - b^5$. For positive integers n, let S_n be the set of the first n primes. Compute

$$\lim_{n\to\infty} \frac{1}{n} \sum_{p \in S_n} f(p)^2.$$

Team Name: ____________________ Answer: ____________________

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