NYCMT 2025-2026 Homework #7

NYCMT

February 6, 2026 - March 6, 2026

These problems are due March 6th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve.

Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on March 6th, you can scan your solutions and email them to captains@nycmathteam.org. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or use one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

Problem 1. Let S_0 be an infinite geometric sequence with first term 1 and sum of elements 8. Given S_n, we define S_{n+1} to be the sequence of the positive differences between consecutive elements of S_n. For example, if S_3 starts with the elements $1, 2, 4, 7$ then S_4 starts with the elements $1, 2, 3$. Compute the sum of the elements of S_{100}.

Problem 2. Let a *good shout* be an integer of the form $p!$ for prime p.

(a) Prove that every positive rational number can be written as a product and quotient of (not necessarily distinct) good shouts. For example, we have the following representation:

$$\frac{6}{7} = \frac{3! \cdot 3! \cdot 5!}{7!}.$$

(a) Prove that this representation is unique up to permutation and simplification of the numerator and denominator. For example, the following representations are considered equivalent:

$$\frac{3! \cdot 3! \cdot 5!}{7!} = \frac{3! \cdot 5! \cdot 3!}{7!} = \frac{2! \cdot 3! \cdot 3! \cdot 5!}{2! \cdot 7!}.$$

Problem 3. Let $ABCDEF$ be a convex hexagon in which $AB = AF$, $BC = CD$, $DE = EF$ and $\angle ABC = \angle EFA = 90°$. Prove that $AD \perp CE$.

Problem 4. Let a, b, c, d be four real numbers such that $a + b + c + d = 20$ and $ab + bc + cd + da = 16$. Find the maximum possible value of $abc + bcd + cda + dab$.

Problem 5. In a 7×7 grid, some squares are colored green such that no grid-aligned rectangle has four distinct corner squares each colored green. Find the maximum possible number of green squares.

Problem 6. [Bonus ⧫]: An polynomial $P(x)$ with integer coefficients has the property that $P(n)$ is squarefree for all integers n. Prove that it is constant.

(An integer is said to be *squarefree* if it is not divisible by p^2 for any prime p. Squarefree integers need not be positive.)