NYCMT 2025-2026 Homework #5

NYCMT

December 5, 2025 - January 9, 2026

These problems are due January 9th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve.

Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on January 9th, you can scan your solutions and email them to jamespapaelias@gmail.com, kylewu32@gmail.com, and sjschool26@gmail.com. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or use one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

Problem 1. Max has a necklace with 7 beads, each of which he paints either red or blue. If rotations of the same pattern of colors are considered the same, in how many ways can Max color his necklace?

Problem 2. Let ABC be an isosceles right triangle with $AB = AC$, and denote by M the midpoint of $\overline{BC}$. Construct rectangle $AXBY$, with X in the interior of $\triangle ABC$, such that $YM = 8$ and $AY^3 + BY^3 = 10^3$. What is the area of quadrilateral $AXBY$?

Problem 3. How many six-digit multiples of 27 have only 3, 6, or 9 as their digits?

Problem 4. Let S be the smallest subset of the integers with the property that $0 \in S$ and for any $x \in S$, we have $3x \in S$ and $3x+1 \in S$. Determine the number of non-negative integers in S less than 2025.

Problem 5. There exists a unique integer polynomial $P(x)$ of degree at most 98 such that each of the coefficients of $P(x)$ is between 0 and 100 inclusive, and the equation

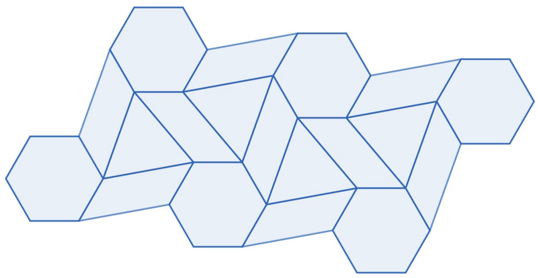
$$(x^2 + x - 2)P(x) \equiv 1 \pmod{101}$$

holds for every $x \in \{0, 1, \ldots, 100\}$ satisfying $x^2 + x - 2 \not\equiv 0 \pmod{101}$. Find the x^2 coefficient of $P(x)$.

Problem 6. Let $f : \mathbb{Z}_{\geq 0} \to \mathbb{Z}$ be a strictly increasing function with $f(0) = 0$, $f(10^3) = 10^6$, and $f(1) + f(2) + \cdots + f(10^3 - 1) = 10^8$. Suppose f is extended to a strictly increasing continuous function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with inverse function h. Compute

$$\left\lfloor \frac{h(1) + h(2) + h(3) + \cdots + h(10^6 - 1)}{10^6} \right\rfloor.$$

Problem 7. The plane is tiled with congruent copies of an equilateral triangle J, a parallelogram K, and a regular hexagon S, by extending the pattern shown below. For each shape, its copies cover $\frac{1}{3}$ of the plane (in the sense that as increasingly large disks in the plane are taken, the fraction of the disk's area covered by each type of shape approaches $\frac{1}{3}$). Determine the angles and the ratio of side lengths in K.



Problem 8. Let $f(x)$ be a quadratic polynomial defined by $f(x) = x^2 + nx$ for positive integer n. We say that $f(x)$ is *second generation* if there exists integer m such that the equation $f(f(x) - m) = m$ has exactly three distinct real solutions. Find the sum of all $n < 100$ such that $f(x)$ is second generation.

Problem 9. [Bonus ⧫]: If you solved problem 9 on the EDAMAME, you discovered the fact that

$$\frac{1}{3541} = 0.\overline{00028240609997175939}.$$

Some people asked us if the digits 999 appearing there in the expansion were coincidental.

(a) Try solving the original problem: If you haven't figured out how to derive the answer above, now would be a good time to do so. (There's no need to submit anything for this part.)

Given that p is a prime number and

$$\frac{1}{p} = 0.\overline{0002a2406099971759b9}$$

for some unknown digits a and b, compute p.

(b) Explain why these digits are not a coincidence.

(c) Generalize: Prove that for any positive integer N, there exists a prime number p such that the decimal expansion of $\frac{1}{p}$ has at least N consecutive 9's somewhere, and an even length repetend.

(d) Is it true that for any positive integer N, there exists a prime number p such that the decimal expansion of $\frac{1}{p}$ has exactly N consecutive 9's somewhere, and an even length repetend?

This part is extremely difficult and is an extra-bonus problem. We encourage sending any partial progress or reductions you make. On this part (only) you may use internet sources and code provided you give citation and your written code respectively if you use any.