HMMT November 2018

November 10, 2018

Team Round

1. [**15**] Four standard six-sided dice are rolled. Find the probability that, for each pair of dice, the product of the two numbers rolled on those dice is a multiple of 4.

2. [**20**] Alice starts with the number 0. She can apply 100 operations on her number. In each operation, she can either add 1 to her number, or square her number. After applying all operations, her score is the minimum distance from her number to any perfect square. What is the maximum score she can attain?

3. [**25**] For how many positive integers $n \leq 100$ is it true that $10n$ has exactly three times as many positive divisors as n has?

4. [**30**] Let a and b be real numbers greater than 1 such that $ab = 100$. The maximum possible value of $a^{(\log_{10} b)^2}$ can be written in the form 10^x for some real number x. Find x.

5. [**35**] Find the sum of all positive integers n such that $1 + 2 + \cdots + n$ divides

$$15[(n+1)^2 + (n+2)^2 + \cdots + (2n)^2].$$

6. [**45**] Triangle $\triangle PQR$, with $PQ = PR = 5$ and $QR = 6$, is inscribed in circle ω. Compute the radius of the circle with center on $\overline{QR}$ which is tangent to both ω and $\overline{PQ}$.

7. [**50**] A 5×5 grid of squares is filled with integers. Call a rectangle *corner-odd* if its sides are grid lines and the sum of the integers in its four corners is an odd number. What is the maximum possible number of corner-odd rectangles within the grid?

   Note: A rectangle must have four distinct corners to be considered *corner-odd*; i.e. no $1 \times k$ rectangle can be *corner-odd* for any positive integer k.

8. [**55**] Tessa has a unit cube, on which each vertex is labeled by a distinct integer between 1 and 8 inclusive. She also has a deck of 8 cards, 4 of which are black and 4 of which are white. At each step she draws a card from the deck, and

   - if the card is black, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance 1 away from this vertex;
   - if the card is white, she simultaneously replaces the number on each vertex by the sum of the three numbers on vertices that are distance $\sqrt{2}$ away from this vertex.

   When Tessa finishes drawing all cards of the deck, what is the maximum possible value of a number that is on the cube?

9. [**60**] Let A, B, C be points in that order along a line, such that $AB = 20$ and $BC = 18$. Let ω be a circle of nonzero radius centered at B, and let ℓ_1 and ℓ_2 be tangents to ω through A and C, respectively. Let K be the intersection of ℓ_1 and ℓ_2. Let X lie on segment $\overline{KA}$ and Y lie on segment $\overline{KC}$ such that $XY \parallel BC$ and XY is tangent to ω. What is the largest possible integer length for XY?

10. [**65**] David and Evan are playing a game. Evan thinks of a positive integer N between 1 and 59, inclusive, and David tries to guess it. Each time David makes a guess, Evan will tell him whether the guess is greater than, equal to, or less than N. David wants to devise a strategy that will guarantee that he knows N in five guesses. In David's strategy, each guess will be determined only by Evan's responses to any previous guesses (the first guess will always be the same), and David will only guess a number which satisfies each of Evan's responses. How many such strategies are there?

    Note: David need not guess N within his five guesses; he just needs to know what N is after five guesses.