1. **[6]** Quincy writes the numbers 1, 3, 4, 8, and 9 on a chalkboard. Every minute, he replaces two numbers m and n on the chalkboard with $5m + n$. Compute the maximum possible value of the final number on the chalkboard.

Proposed by Gabe Levin

Solution. Note that we are just expressing a permutation of the given numbers as a "base-5" numeral, without the restriction that digits are less than 5. Thus we want the digits to be in decreasing order, so the answer is $9 \cdot 5^4 + 8 \cdot 5^3 + 4 \cdot 5^2 + 3 \cdot 5 + 1 = \boxed{6741}$.

2. **[6]** Compute the number of positive integers $n < 50$ such that both n and $50 - n$ have an even number of divisors.

Proposed by Gabe Levin

Solution. We are just counting the number of positive integers n such that n and $50 - n$ are both not squares, so we use complementary counting. For any perfect square less than 50, we can choose to put it in the first coordinate of the pair or the second, giving 14 pairs. However, the pairs $(1, 49)$, $(49, 1)$, and $(25, 25)$ are counted twice, so there are only 11 pairs. Thus, the answer is $49 - 11 = \boxed{38}$.

3. **[7]** Percy rolls eight fair six-sided dice and records their values a, b, c, d, e, f, g, and h. If P is the probability that

$$(a + b)^{c+d}(e + f)^{g+h} = 2025,$$

and $P = p^a q^b r^c$ where p, q, r are distinct primes and a, b, c are nonzero integers, compute $pa + qb + rc$.

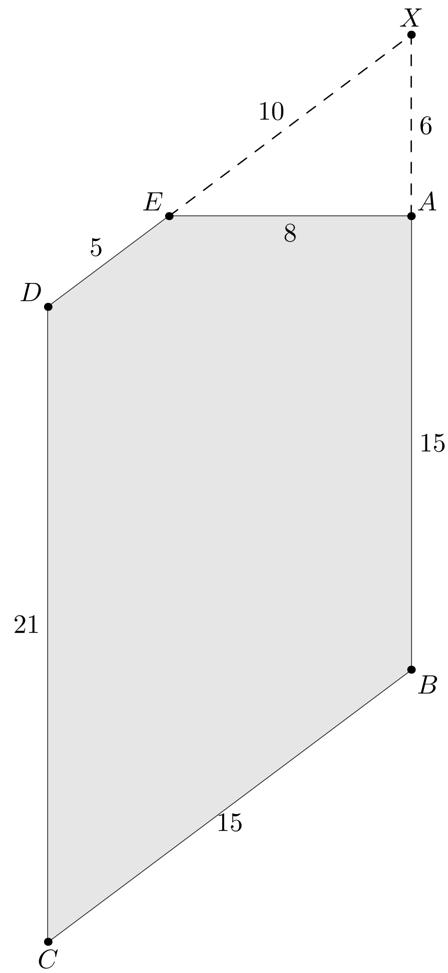
Proposed by Tanvir Ahmed

Solution. We can either express 2025 as $9^2 5^2$ or as $3^4 5^2$. Suppose $e + f = 5$ and $g + h = 2$. Then, we have 4 choices for (e, f) and 1 choice for (g, h), so 4 choices for (e, f, g, h). If $a + b = 9$ and $c + d = 2$, then we have a total of 4 choices for (a, b, c, d), while if $a + b = 3$ and $c + d = 4$, we have 2 choices for (a, b) and 3 for (c, d), for a total of 6 for (a, b, c, d). We may also swap (a, b, c, d) and (e, f, g, h), so there are a total of $2 \cdot 4 \cdot (4 + 6) = 80$ valid octuples. Thus, $P = 80/6^8 = 5^1 2^{-4} 3^{-8}$, for an answer of $5 - 8 - 24 = \boxed{-27}$.

4. **[7]** Let $ABCDE$ be a pentagon and suppose $AB \parallel CD$ and $BC \parallel DE$. Compute the area of this pentagon given that $AB = BC = 15$, $CD = 21$, $DE = 5$, and $EA = 8$.

Proposed by Gabe Levin

Solution. Extend DE and AB to meet at X so that $XBCD$ is a parallelogram:



Observe that XAE is a right triangle with side lengths 6, 8, and 10. Thus EA is 2/3 the height of the parallelogram, so that $[XBCD] = 12 \cdot 21 = 252$. The area of XAE is 24, so $[ABCDE] = 252 - 24 = \boxed{228}$.

5. **[8]** Let $x = 1252^3 - 1248^3$. Compute $\sqrt{x - 16}$.

Proposed by Gabe Levin

Solution. If $t = 1250$, then we are computing $\sqrt{(t+2)^3 - (t-2)^3 - 16} = \sqrt{12t^2 + 16 - 16} = 2t\sqrt{3} = \boxed{2500\sqrt{3}}$.

6. **[8]** Compute

$$\sum_{m|1024} \sum_{n|1024} \left\lfloor \frac{m}{n} \right\rfloor.$$

Proposed by Gabe Levin

Solution. Note that for each k, $\lfloor \frac{m}{n} \rfloor = 2^k$ whenever $n = 2^r$ and $m = 2^{k+r}$, so that there are $11 - k$ choices for m and n. Thus we are computing

$$S = \sum_{k=0}^{10} 2^k(11 - k).$$

Then

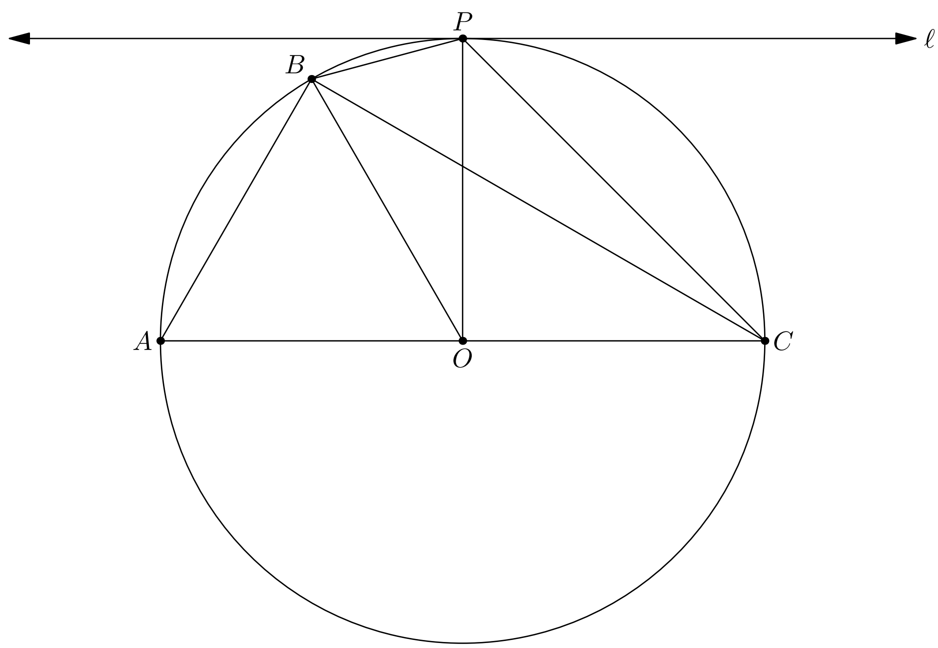
$$S = 2S - S = \sum_{k=1}^{11} 2^k(12 - k) - \sum_{k=0}^{10} 2^k(11 - k) = 2^{11} - 11 + \sum_{k=1}^{10} 2^k,$$

which evaluates to $2 \cdot 2^{11} - 13 = \boxed{4083}$.

7. **[9]** Let ABC be a triangle with $AB = 1$, $BC = \sqrt{3}$, and $CA = 2$. Let O be the circumcircle of ABC and let ℓ be the tangent to O parallel to AC and closer to B. Suppose ℓ intersects O at P. Compute the area of triangle OPB.

Proposed by Gabe Levin

Solution. Shown below is a diagram:



Notice that ABC is a 30-60-90 right triangle, so $\triangle ABO$ is equilateral, and thus $\angle AOB = 60^\circ$, Furthermore, since AC is a diameter of the circle, we must have that $PO \perp BC$, so $\angle COP = 90^\circ$. Thus, $\angle BOP = 30^\circ$, so $[OPB] = \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 30^\circ = \boxed{\frac{1}{4}}$.

8. **[9]** Call a set $S \subset \{1, 2, \dots, 2025\}$ *corwizzy* if for any $a, b \in S$ with $ab \leq 2025$, $ab \in S$. Compute the smallest positive intege r $n > 1$ such that there exists a corwizzy set S whose elements sum to n.

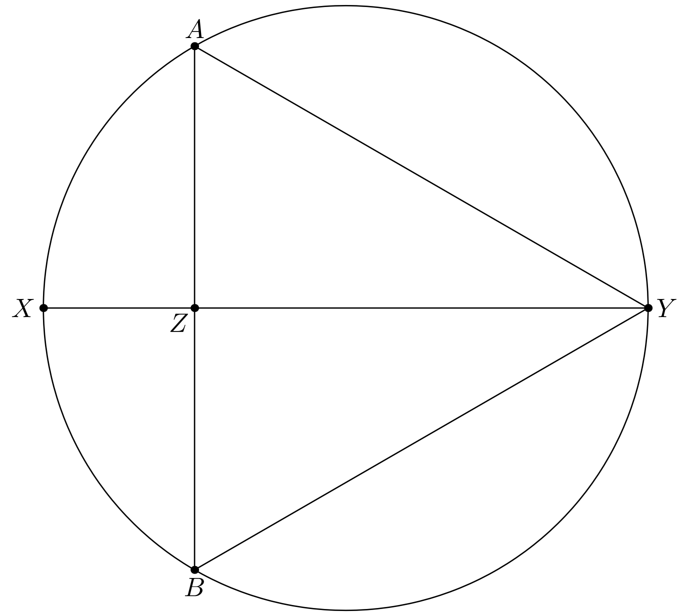
Proposed by Gabe Levin

Solution. Note that any corwizzy set S has an element greater than 45, as otherwise we have $(\max S)^2 \in S$, a contradiction (since $n > 1$). Thus the answer is at least $\boxed{46}$, and clearly $\{46\}$ is corwizzy.

9. **[10]** Let O be a circle and $XY = 8$ be a diameter of O. Let A lie on O and let the circle with center X and radius XA intersect O at $B \neq A$. Compute the maximum possible area of triangle ABY.

Proposed by Gabe Levin

Solution. Clearly, $AXBY$ forms a kite, so B is just the reflection of A over XY. Let Z be the foot of the altitude from A to XY, and let $x = ZY$.



Thus, $XZ = 8 - x$. Since AXY and AZ is the altitude to the hypotenuse, $AZ = \sqrt{XZ \cdot YZ} = \sqrt{x(8-x)}$. We are trying to maximize $[ABY] = AZ \cdot YZ = x\sqrt{x(8-x)}$. It suffices to maximize the square of this quantity instead, which is $x^3(8-x)$. It suffices to maximize $\frac{1}{27}$ times this quantity, which is

$$\frac{x^3(8-x)}{27} = \left(\frac{x}{3}\right)^3(8-x) \le \left(\frac{3 \cdot \frac{x}{3} + 8 - x}{4}\right)^4 = 16$$

by AM-GM, with equality when $\frac{x}{3} = 8 - x$. This implies that $x = 6$ is optimal, in which case $[ABY] = 6\sqrt{6 \cdot 2} = \boxed{12\sqrt{3}}$.

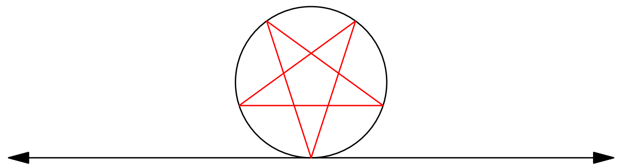
10. **[Up to 10]** Submit an ordered quadruple of integers (a, b, c, d) with $0 \le a, b, c, d \le 50$. Let $\mathcal{R}$ be the axis-aligned rectangle who's bottom left vertex has coordinates (a, b) and who's upper right vertex has coordinates (c, d). If $\mathcal{R}$ coincides with another teams rectangle or if $\mathcal{R}$ has non-zero intersection with another teams rectangle and neither contain each other, you get 0 points. Otherwise, you get $\frac{|b-a||d-c|}{250}$ points.

Proposed by Lucas Pavlov

11. **[11]** Given a circular mirror, at how many angles strictly above the horizontal from the south pole can a laser be shot such that it bounces at most 20 times before returning?

Proposed by Lucas Pavlov

Solution. Here is an example of the ball bouncing 4 times and returning.



We first consider the number for the laser to bounce exactly n times before returning. Let θ be the angle of the arc formed by the south pole and the point at which the laser is shot in radians. Then, including the spot at which the laser is shot at, the laser will divide the circle into $n+1$ equally sized arcs. Thus, $(n+1)\theta$ must be a multiple of 2π, but $k\theta$ can't be a multiple of 2π for any $k \le n+1$. Therefore, $\theta = \frac{2\pi j}{n+1}$ for some positive integer $j \le n$, and the other condition is equivalent to $\gcd(j, n) = 1$ because otherwise, we could reduce this fraction. Thus, there are $\varphi(n+1)$ ways to bounce the laser such that it returns after exactly n bounces. The final answer is thus

$$\sum_{n=2}^{21} \varphi(n) = \boxed{139}.$$

12. **[11]** What is the largest positive integer k such that there exist k consecutive four-digit positive integers, each with no more than 3 distinct digits?

Proposed by Gabe Levin

Solution. Clearly the set $\{9877, 9878, \ldots, 9999\}$, with size $\boxed{123}$, is such a set of integers, as the largest four-digit integer with four distinct digits is 9876. To show maximality, observe that, if a set contains elements with different first digits, say n and $n+1$, the minimal element of the set is at least $\overline{n976}$ (unless $n = 9$, in which case $n+1 = 10$, which is too large). The maximal element of the set is $\overline{(n+1)013}$, for a set size clearly less than 123. Thus any such set has all elements starting with the same digit, say n. Clearly, then, it is optimal for the set to contain all integers of the form $\overline{nnxy}$. If $n \neq 9$, then the set cannot contain both of $\overline{n(n-1)98}$ and $\overline{n(n-1)96}$, as it only contains the former if $n = 8$. Moreover, the set cannot contain both of $\overline{n(n+1)01}$ and $\overline{n(n+1)03}$, as it only contains the former if $n = 1$. Thus the maximum length of the set is 108. If $n = 9$, we can clearly see that the set cannot contain 9876, so the set listed above is the largest such set.

13. **[12]** An 8×9 grid is filled with the 72 divisors of 36000, randomly and without replacement. CrussoCode randomly and uniformly picks one of the squares, and computes its prime factorization along with the prime factorizations of its neighbors (only lateral neighbors, not diagonal). What is the probability that each such prime factorization has pairwise distinct exponents for each prime factor of 36000?

(For example, $2^3 3^3$ and $2^4 3^2$ have pairwise distinct exponents as $3 \neq 4$ and $3 \neq 2$, but $2^4 3^2$ and $2^3 3^2$ do not, as $2 = 2$.)

Proposed by Gabe Levin

Solution. Note that $36000 = 2^5 3^2 5^3$. For any chosen square that is not a corner, the set containing the square and its neighbors has at least four elements. However, there are only three possible exponents for the prime 3. Thus the square must be a corner. We then count the number of possible sets of exponents for each of the three prime factors, which are $6 \cdot 5 \cdot 4$, $3 \cdot 2 \cdot 1$, and $4 \cdot 3 \cdot 2$. Thus, the answer is

$$\frac{4}{72} \cdot \frac{(6 \cdot 5 \cdot 4)(3 \cdot 2 \cdot 1)(4 \cdot 3 \cdot 2)}{72 \cdot 71 \cdot 70} = \boxed{\frac{4}{1491}}.$$

14. **[12]** For how many ordered triples of integers (a, b, c) satisfying $0 \le a, b, c \le 20$ do there exist integers $(x, y, z) \neq (0, 0, 0)$ satisfying

$$ax = by + cz$$
$$bx = cy + az$$
$$cx = ay + bz?$$

Proposed by Gabe Levin

Solution. Adding the equations, we see $(a+b+c)x = (a+b+c)(y+z)$. If $(a, b, c) \neq (0, 0, 0)$, then $x = y+z$. Thus we have $ay + az = by + cz$ and $by + bz = cy + az$. Thus, $(a - b)y = (c - a)z$ and $(b - c)y = (a - b)z$. WLOG $a \le b \le c$. Then we have $(a - b)^2 y = (c - a)(a - b)z = (c - a)(b - c)y$. Looking at signs, we see $(a - b)^2$ is nonnegative, while $(c - a)(b - c)$ is nonpositive, so that $a = b = c$, giving $\boxed{21}$ solutions. (If $y = 0$, we have $bc = a^2$, again implying $a = b = c$.)

Solution. We can rewrite this system of equations as

$$ax - by - cz = 0$$
$$bx - cy - az = 0$$
$$cx - ay - bz = 0.$$

$(0, 0, 0)$ is always a solution to this system, so for there to be any other solutions, the determinant of the coefficient matrix must be 0. Notice that

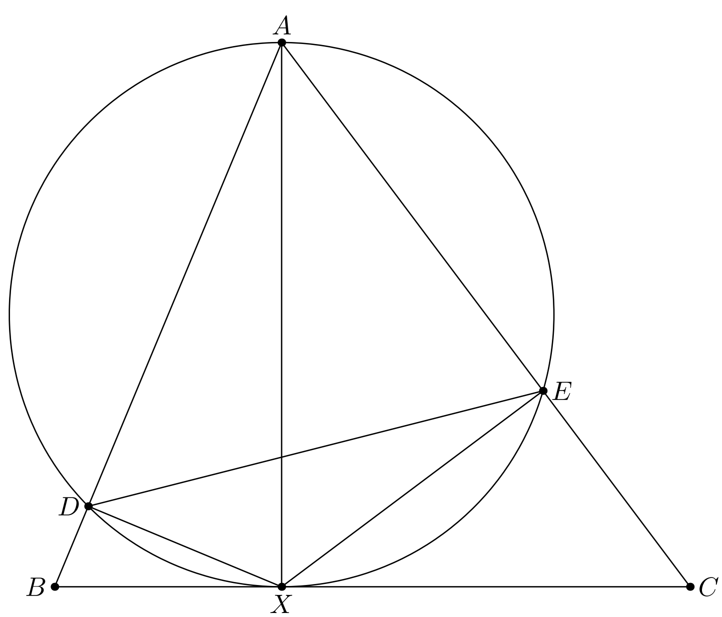
$$\begin{vmatrix} a & -b & -c \\ b & -c & -a \\ c & -a & -b \end{vmatrix} = 3abc - a^3 - b^3 - c^3 = -(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc).$$

Thus, $a + b + c = 0$ or $a^2 + b^2 + c^2 = ab + ac + bc$. The first case corresponds to $(a, b, c) = (0, 0, 0)$, and note that by weighted AM-GM (in particular, adding up the inequality $a^2 + b^2 \ge 2ab$ cyclically) or Muirhead's inequality, the second case only holds if $a = b = c$. Either way, there are nonzero solutions exist if and only if $a = b = c$, and we get an answer of $\boxed{21}$ as in the previous solution.

15. **[13]** Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. Let X be the foot of A onto BC and let D and E be the feet of X onto AB and CA respectively. Compute DE.

Proposed by Gabe Levin

Solution. Here is a diagram:



Note that $ADXE$ is cyclic with diameter AX as it contains two opposite right angles. The radius of this circumcircle is $\frac{1}{2}AX$. $[ABC]$ is well known to be 84 for a 13-14-15 triangle, so $AX = 2 \cdot \frac{84}{14} = 12$, and thus the radius of the circle is 6. In a similar fashion, we can find that the altitude from B to AC is $\frac{168}{15}$, so

$$\sin A = \frac{168/15}{AB} = \frac{168}{13 \cdot 15}.$$

Finally, by the Law of Sines,

$$DE = 2R \sin A = 12 \cdot \frac{168}{13 \cdot 15} = \boxed{\frac{672}{65}}.$$

16. **[13]** Let a_n, b_n be sequences with $a_1 = 6$, $b_1 = 7$ and

$$a_{n+1} = {b_n}^2 + a_n b_n$$
$$b_{n+1} = {a_n}^2 + a_n b_n$$

for all $n \geq 1$. Compute the number of divisors of a_{10}.

Proposed by Corwin Eisenbeiss

Solution. Let $x_n = a_n + b_n$ and $y_n = a_n - b_n$ for all n. Adding and subtracting the given equations yields

$$a_{n+1} + b_{n+1} = {a_n}^2 + 2a_n b_n + {b_n}^2$$
$$a_{n+1} - b_{n+1} = {b_n}^2 - {a_n}^2$$

which in the context of our new sequences means

$$x_{n+1} = {x_n}^2$$
$$y_{n+1} = -x_n y_n.$$

Since $x_1 = a_1 + b_1 = 13$, $x_n = 13^{2^{n-1}}$ so $x_{10} = 13^{2^9} = 13 \cdot 13^{2^9-1}$. Moreover, $y_1 = -1$, so y_{10} is positive, meaning $y_{10} = x_9 x_8 \cdots x_{10} = 13^{2^8+2^7+\cdots+2^0} = 13^{2^9-1}$. Moreover, $a_{10} = \frac{x_{10}+y_{10}}{2} = 7 \cdot 13^{2^9-1} = 7^1 \cdot 13^{511}$ so a_{10} has $(1+1)(511+1) = \boxed{1024}$ divisors.

17. **[14]** Mario starts at $(0,0,0)$ and is trying to reach Peach's CASTLE at $(41,6,7)$. Every time he moves, Mario can do one of the following:

(i) Move $+2$ units in the x direction

(ii) Choose two of the x, y, z-directions and move $+1$ units in both of them

Let N be the number of distinct paths that Mario can take to Peach's CASTLE. Find the sum of the distinct prime factors of N.

Proposed by Steven Breger

Solution. Every move must increase the sum of Mario's coordinates by 2, so Mario will make $\frac{41+6+7}{2} = 27$ moves in total. We can first choose 6 moves in which Mario will move in the y direction and then 7 moves in which he will move in the z direction. The moves that we choose for the y and z directions may overlap. Then, For all moves that don't have 2 directions already determined, we make Mario move in the x direction. Thus, $N = \binom{27}{6}\binom{27}{7} = 2^4 \cdot 3^6 \cdot 5 \cdot 11^2 \cdot 13^2 \cdot 23$, which gives an answer of $2+3+5+11+13+23 = \boxed{57}$.

Solution. We construct a three-variable generating function to represent Mario's movement. In particular, we consider $(x^2 + xy + xz + yz)^{27}$. Each term of $x^2 + xy + xz + yz$ represents one of the possible moves Mario can make, and we raise it to the 27th power since Mario makes 27 moves. Note that this factors as

$$(x+y)^{27}(x+z)^{27}.$$

We want to find the coefficient of $x^{41}y^6z^7$. This term can only be attained by multiplying the y^6 term from $(x+y)^{27}$ and the z^7 term from $(x+z)^{27}$. By the binomial theorem, the product of the corresponding coefficients is $\binom{27}{6}\binom{27}{7}$, and we get the same answer as in the previous solution.

18. **[14]** Let f be a function such that $f(p^k) = p^k + p^{k-1}$ for prime p and positive integer k and $f(ab) = f(a)f(b)$ for all relatively prime integers a, b. Find the sum of all possible distinct values of $\frac{f(2024x)}{2024f(x)}$ for integer x.

Proposed by Corwin Eisenbeiss

Solution. Note $2024 = 45^2 - 1^2 = 44 \cdot 46 = 2^3 \cdot 11 \cdot 23$. As such, let $x = mn$ such that all the factors of 2, 11, and 23 are in m so m and n are coprime and similarly $2024m$ and n are coprime. It follows that

$$f(2024x) = f(2024mn) = f(2024m)f(n)$$

and similarly

$$2024f(x) = 2024f(mn) = 2024f(m)f(n)$$

so we have

$$\frac{f(2024x)}{2024f(x)} = \frac{f(2024m)f(n)}{2024f(m)f(n)} = \frac{f(2024m)}{2024f(m)}.$$

If $m = 2^a \cdot 11^b \cdot 23^c$, this expression becomes

$$\frac{f(2024m)}{2024f(m)} = \frac{f(2^{a+3} \cdot 11^{b+1} \cdot 23^{c+1})}{2^3 \cdot 11^1 \cdot 23^1 \cdot f(2^a \cdot 11^b \cdot 23^c)} = \frac{f(2^{a+3})}{2^3 \cdot f(2^a)} \cdot \frac{f(11^{b+1})}{11^1 \cdot f(11^b)} \cdot \frac{f(23^{c+1})}{23^1 \cdot f(23^c)}.$$

Note that $f(p^k) = p^k + p^{k-1} = p^k\left(1 + \frac{1}{p}\right)$ holds for $k \geq 0$ (possible exponents) unless $k = 0$ where it is 1. Thus if $d \geq 0$,

$$\frac{f(p^{k+d})}{p^d \cdot f(p^k)} = \frac{p^{k+d}(1 + \frac{1}{p})}{p^d \cdot p^k(1 + \frac{1}{p})} = 1$$

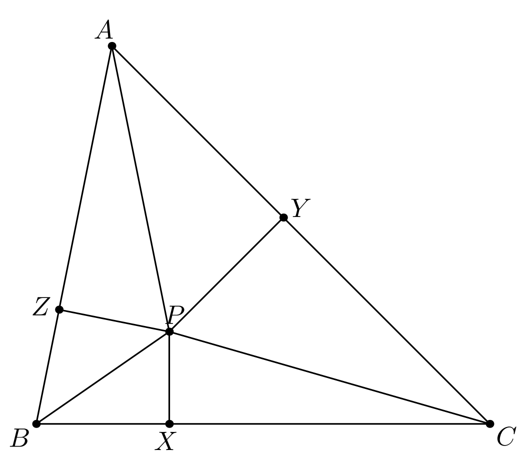
unless $k = 0$ whence $f(p^k) = 1$ and the fraction is $\frac{f(p^d)}{p^d} = 1 + \frac{1}{p}$. Thus, depending on whether m has the prime or not, each $\frac{f(p^{k+d})}{p^d f(p^k)}$ is either 1 or $1 + \frac{1}{p}$. Each possible value made by combining these is distinct as the denominator is not divisible by p if 1 is picked and it is if $1 + \frac{1}{p}$ is picked. As such, we may simply evaluate

$$\left(1 + 1 + \frac{1}{2}\right)\left(1 + 1 + \frac{1}{11}\right)\left(1 + 1 + \frac{1}{23}\right) = \left(\frac{5}{2}\right)\left(\frac{23}{11}\right)\left(\frac{47}{23}\right) = \boxed{\frac{235}{22}}.$$

19. **[15]** Let triangle ABC have $AB = 5$, $BC = 6$, and $CA = 7$ and P be a point in the plane. If X, Y, Z are the feet of the altitudes from P to AB, BC, and CA, find the minimum possible value of $PX^2 + PY^2 + PZ^2$ across all P.

Proposed by Corwin Eisenbeiss

Solution. A diagram is shown below.



The key observation is that $PX \cdot BC = 2[PBC]$ and analogously $PY \cdot CA = 2[PCA]$ and $PZ \cdot AB = 2[PAB]$. Summing these, we get

$$PX \cdot BC + PY \cdot CA + PZ \cdot AB = 2[PBC] + 2[PCA] + 2[PAB] \geq 2[ABC]$$

where the last is really an inequality because P may lie outside the triangle. By the Cauchy-Schwarz inequality, we deduce

$$(PX^2 + PY^2 + PZ^2)(BC^2 + CA^2 + AB^2) \geq (PX \cdot BC + PY \cdot CA + PZ \cdot AB)^2 \geq (2[ABC])^2 \geq 4[ABC]^2$$

with equality if and only if $PX = \lambda BC$, $PY = \lambda CA$, $PZ = \lambda AB$ for some real λ. We may calculate

$$BC^2 + CA^2 + AB^2 = 6^2 + 7^2 + 5^2 = 110$$

This means

$$PX^2 + PY^2 + PZ^2 \geq \frac{4[ABC]^2}{BC^2 + CA^2 + AB^2}.$$

The semiperimeter is $s = 9$, so by Heron's formula

$$[ABC]^2 = 9(9-5)(9-6)(9-7) = 9 \cdot 4 \cdot 3 \cdot 2 = 216.$$

Substituting, this means the minimum is $\frac{4 \cdot 216}{110} = \boxed{\frac{432}{55}}$.

To see that equality may hold, we in essence need to find a point P for which $PX : PY : PZ = BC : CA : AB = 6 : 7 : 5$ or in other words a point P for which $[PBC] : [PCA] : [PAB] = 36 : 49 : 25$. If we take the point C' on AB for which $P = C'$ causes $[PBC] : [PCA] = 36 : 49$, then for all P on CC', the heights of $\triangle CPA$ and $\triangle CC'A$ are the same and similarly for $\triangle CPB$ and $\triangle CC'B$ so $[PBC] : [PCA] = 36 : 49$, meaning all points on a cevian of the triangle through C satisfy $[PBC] : [PCA] = 36 : 49$. There is similarly a cevian of the triangle through A, all points on which satisfy $[PCA] : [PAB] = 49 : 25$, and so their intersection gives our desired minimum.

20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put Y if you think the statement is true, N if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.

(a) Let ABC be a triangle and A' be the reflection of A over BC. Define B' and C' analogously. It is possible for triangle ABC to be strictly inside of triangle $A'B'C'$.

Proposed by Lucas Pavlov

(b) Given triangle ABC, there always exists a point P in the same plane such that $PA : PB : PC = 41 : 67 : 69$.

Proposed by Steven Breger

(c) Put an infinite, axis-aligned grid on the coordinate plane. Let a *phone* be a contiguous path of squares of the grid (that is, no 2×2 set of squares is completely on the path). Call a phone *new* if it is possible to get from one end of the phone to the other while staying on the squares of the phone and only moving up or to the right. Any new phone can tile the grid.

Proposed by Lucas Pavlov

(d) Let $f(x) = \frac{ax+b}{cx+d}$ and $g(x) = \frac{px+q}{rx+s}$ be functions with real coefficients on the reals such that $f(f(x))$ and $g(g(x))$ are not the identity. Suppose $f(g(x)) = g(f(x))$ for all x. Then $f(x) - x$ and $g(x) - x$ have the same roots.

Proposed by Gabe Levin

(e) The function $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \sin^6\left(x - \frac{3\pi}{8}\right) + \sin^6\left(x - \frac{\pi}{8}\right) + \sin^6\left(x + \frac{\pi}{8}\right) + \sin^6\left(x + \frac{3\pi}{8}\right).$$

For any $x \in \mathbb{R}$, there exists some $y \in \mathbb{R}$ such that $0 < |x - y| < \frac{\pi}{8}$ and $f(x) = f(y)$.

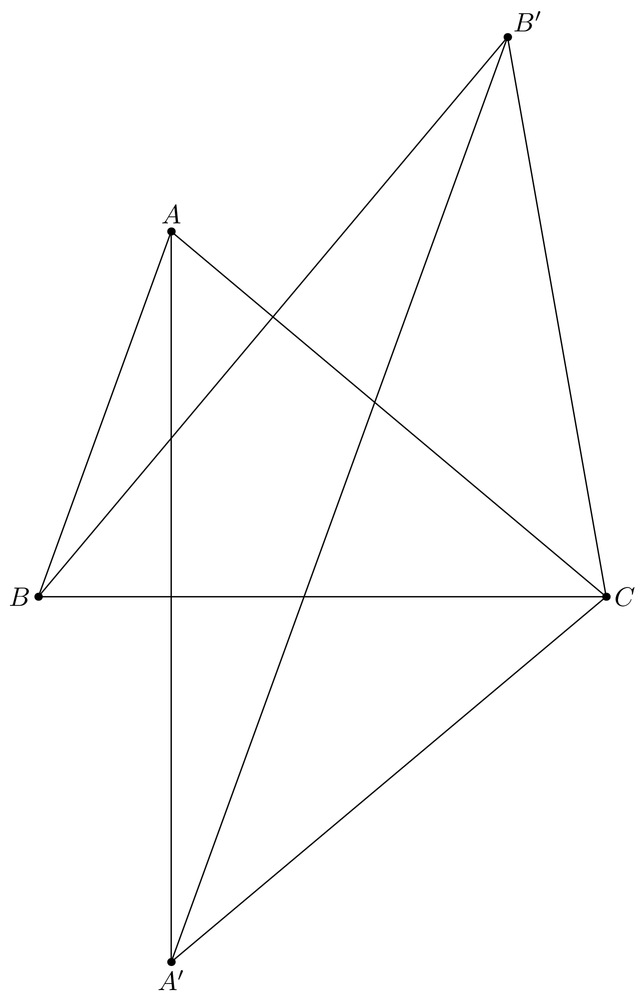
Proposed by Gabe Levin

(f) For all positive integers $n > 1$ and $k < n$, $\gcd(\binom{n}{k}, n) > 1$.

Proposed by Gabe Levin

Solution. $\boxed{NNYNYY}$.

(a) The claim is $\boxed{\text{False}}$. We claim that $A'B'$ intersects AC and BC if and only if $\angle C \le 60^\circ$. Here is a diagram:

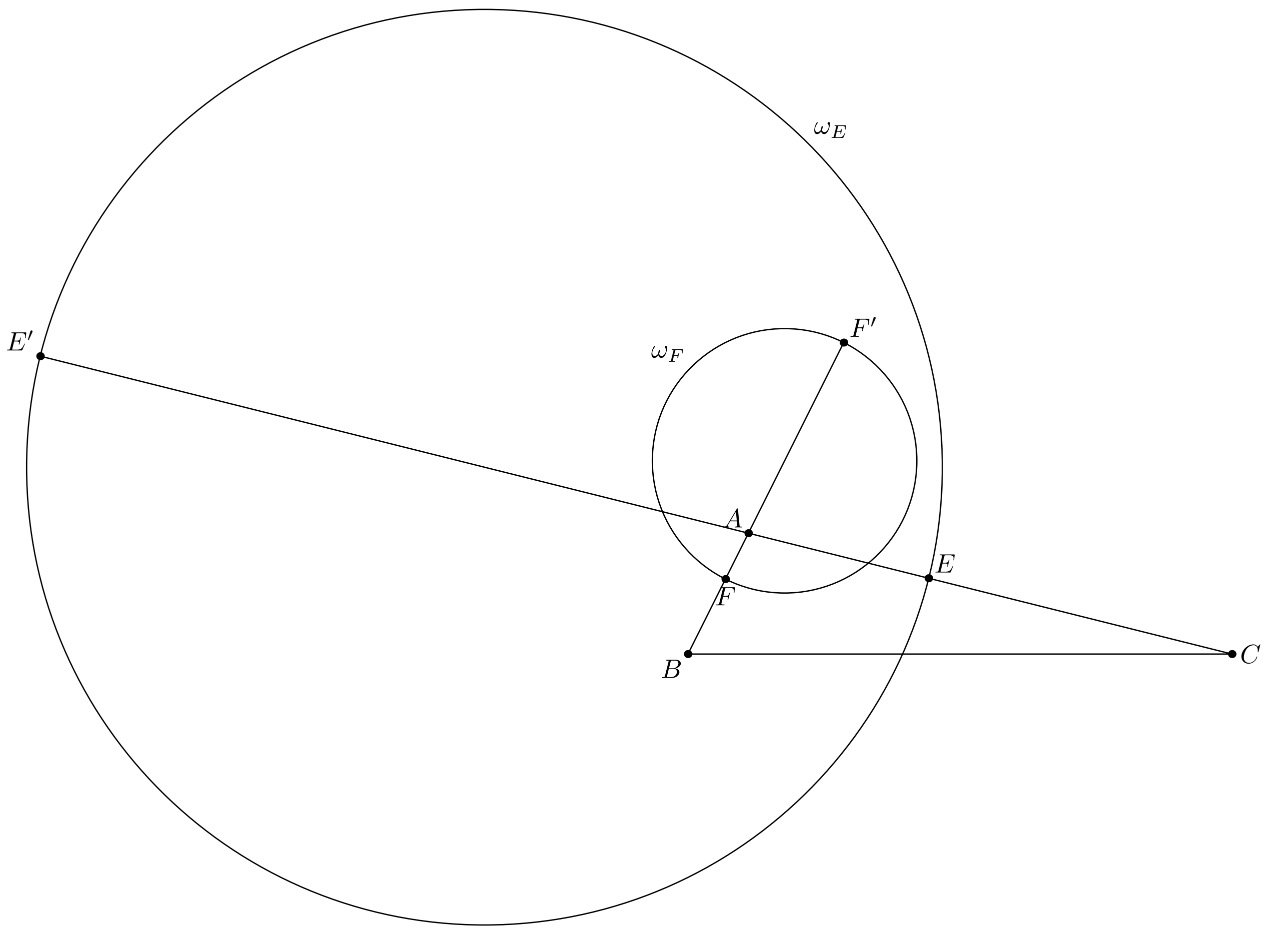


Notice that $A'B'$ intersects AC and BC if and only if the angle $\angle A'CB'$ (particularly the side of the angle that contains segment AB) is less than or equal to 180°. However, this angle is just $\angle A'CB + \angle BCA + \angle ACB' = 3\angle C$, so this condition is equivalent to $\angle C \le 60^\circ$. $\triangle ABC$ must have an angle less than or equal to 60°, so at least one of the sides of $A'B'C'$ intersect ABC, which means it can't be contained strictly inside ABC.

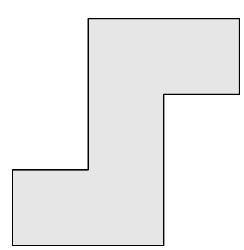
(b) The claim is $\boxed{\text{False}}$. Define $S = \{P \mid PA : PB = 41 : 67\}$ and define $T = \{P \mid PA : PC = 41 : 69\}$. Note that for some point P, if $P \in S$ and $P \in T$, then $PA : PB : PC = 41 : 67 : 69$. We are thus left to prove that $S \cap T$ can be empty for some ABC.

Let F, F' be the two points on line AB that divide AB into a ratio of $41 : 67$. Let E and E' be the two points on line AC that divide AC into a ratio of $41 : 69$. Let ω_F be the circle with diameter FF' and let ω_E be the circle with diameter EE'. Let X be a point on ω_F and Y be a point on ω_E. Then, by the angle bisector theorem, ω_F is the Apollonian circle of $\triangle ABX$ while ω_E is the Apollonian circle of $\triangle ACY$. Thus, $S = \omega_F$ and $T = \omega_E$. All that is left to do is to construct ABC such that ω_F and ω_E do not intersect.

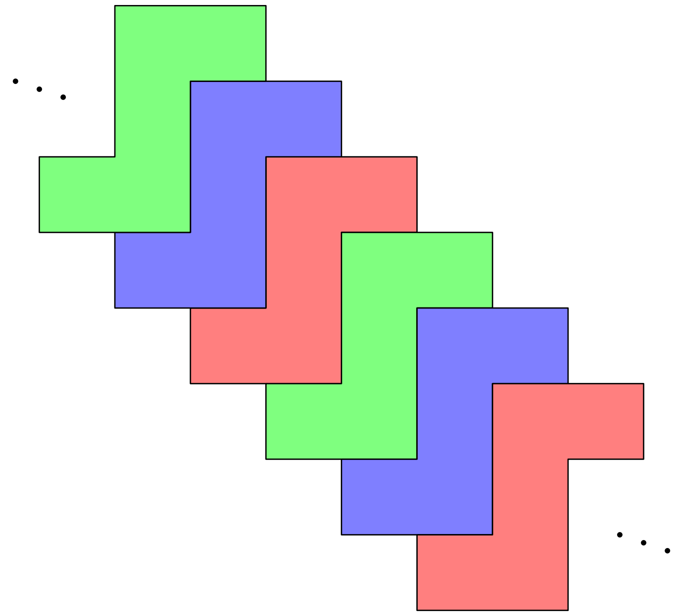
Imagine fixing segment AB and moving point C far to the side. ω_F would not change or move while ω_E would grow large leaving ω_F contained completely inside it. This is pictured in the diagram below. There are, in fact, many ABC where these circles do not intersect.



(c) The claim is True. By taking a new phone, copying it, and moving it up one square up and one square left or one square right and one square down, we can create diagonal strips of squares in the plane. For example, suppose we have the following new phone:



We can tile this new phone to form diagonal strips as follows:



These diagonal strips can be tiled together to cover the entire plane.

(d) The claim is $\boxed{\text{False}}$. Let $f(x) = \frac{x-2}{x+4}$ and let $g(x) = -1$. Then $g(f(x)) = -1$ and $f(g(x)) = f(-1) = \frac{-3}{3} = -1$. Thus $f(g(x)) = g(f(x))$. Moreover, $g(g(2)) = -1 \neq 2$ and $f(f(2)) = f(0) = -\frac{1}{2}$, so that f^2 and g^2 are not the identity. However, $f(x) - x$ has roots -1 and -2, while $g(x)$ only has a root at -1.

The claim *is* true, however, if we add the additional constraint that f and g are nonconstant. We leave the proof as an exercise.

(e) The claim is $\boxed{\text{True}}$. We prove the stronger claim that f is constant. We redefine f by shifting down $\frac{\pi}{8}$ to give an easier formulation of

$$f(x) = \cos^6(x) + \sin^6\left(x - \frac{\pi}{4}\right) + \sin^6(x) + \sin^6\left(x + \frac{\pi}{4}\right).$$

If we let $s = \sin x$ and $t = \cos x$, note that $\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(s-t)$ while $\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(s+t)$. Thus

$$\begin{aligned} f(x) &= s^6 + t^6 + \frac{1}{8}\left((s-t)^6 + (s+t)^6\right) \\ &= \frac{1}{8}\left((s-t)^6 + 8(s^6+t^6) + (s+t)^6\right) \\ &= \frac{1}{8}\left(10s^6 + 30s^4t^2 + 30s^2t^4 + 10t^6\right) \\ &= \frac{5}{4}(s^2+t^2)^3 = \frac{5}{4}, \end{aligned}$$

as desired.

(f) The claim is $\boxed{\text{True}}$. Let $p \mid n$ and $v = \nu_p(n)$ and suppose $p \nmid \binom{n}{k}$ for some k. By Lucas' theorem,

$$\binom{n}{k} \equiv \prod_i \binom{n_i}{k_i} \not\equiv 0 \pmod{p},$$

where n_i and k_i are the ith digits in the base p representations of n and k respectively. Then note that $n_{v-1} = n_{v-2} = \cdots = n_0 = 0$. Thus $k_{v-1} = k_{v-2} = \cdots = k_0 = 0$, so that $p^v \mid k$. Thus if $\gcd\left(n, \binom{n}{k}\right) = 1$, $p^{\nu_p(n)} \mid k$ for each $p \mid n$. Thus $n \mid k$, as desired.

21. **[16]** For positive integers n and k, let $\psi_k(n)$ be the smallest positive integer such that $n\psi_k(n)$ is a perfect kth power. Call a positive integer n *psichotic* if there exists a square-free positive integer N such that

$$n = \sum_{d \mid N^{104}} \varphi(\psi_3(d))\varphi(\psi_5(d))\varphi(\psi_7(d)).$$

Find the number of positive integers $n \le 2025^6$ that are psichotic (recall that for positive integers n, $\varphi(n)$ is the number of integers relatively prime to n between 1 and n inclusive).

Proposed by Tanvir Ahmed

Solution. Let

$$f(N) = \sum_{d \mid N^{104}} \varphi(\psi_3(d))\varphi(\psi_5(d))\varphi(\psi_7(d)).$$

We first evaluate $f(p)$ for primes p. The sum ranges over all divisors $d = p^k$ of p^{104}. Since $0 \le k < 105$, k can also be uniquely described by its residue mod 3, 5, and 7 by the Chinese Remainder Theorem. If $k \equiv -a \pmod 3$, $k \equiv -b \pmod 5$, and $k \equiv -c \pmod 7$, where $0 \le a < 3$, $0 \le b < 5$, and $0 \le c < 7$ then $\psi_3(k) = p^a$, $\psi_5(k) = p^b$, and $\psi_7(k) = p^c$. Thus,

$$\begin{aligned} f(p) &= \left(\sum_{d \mid N^2} \varphi(\psi_3(d))\right)\left(\sum_{d \mid N^4} \varphi(\psi_5(d))\right)\left(\sum_{d \mid N^6} \varphi(\psi_7(d))\right) \\ &= (\varphi(1) + \varphi(p) + \varphi(p^2))(\varphi(1) + \cdots + \varphi(p^4))(\varphi(1) + \cdots + \varphi(p^6)) \end{aligned}$$

since after expanding the parentheses, the term corresponding to every divisor of p^{104} is counted exactly once, by our CRT argument. Note that the above product gives us $f(p) = p^2 \cdot p^4 \cdot p^6 = p^{12}$.

Now we prove that f is multiplicative. First of all, ψ_k is clearly multiplicative since it acts on each distinct prime divisor of a number independently, and for $\gcd(m, n) = 1$, all prime factors of $\psi_k(m)$ are prime factors of m and analogously for $\psi_k(n)$, so $\gcd(\psi_k(m), \psi_k(n)) = 1$. Therefore, since φ is multiplicative,

$$\varphi(\psi_k(mn)) = \varphi(\psi_k(m)\psi_k(n)) = \varphi(\psi_k(m))\varphi(\psi_k(n)),$$

so $\varphi(\psi_k(N))$ is multiplicative. Thus, $\varphi(\psi_3(N))\varphi(\psi_5(N))\varphi(\psi_7(N))$ is also multiplicative. $f(N)$ is just the sum of this function over all divisors of N^{104}, which is also multiplicative. Thus, for all square-free N, $f(N) = N^{12}$. The problem thus reduces to counting the number of square-free positive integers whose 12th power is less than or equal to 2025^6, which is equivalent to counting the number of square-free positive integers less than or equal to $\sqrt[12]{2025^6} = 45$. It is relatively easy to count that there are $\boxed{29}$ such integers.

22. **[16]** Compute the maximum value of

$$f(x) = \frac{x^2 + x - 2}{x^4 + 2x^3 - x^2 - 2x + 3}$$

as x ranges over $\mathbb{R}$.

Proposed by Gabe Levin

Solution. Let $y = x^2 + x$. Then

$$f(x) = \frac{y-2}{y^2 - 2y + 3} = \left(y + \frac{3}{y-2}\right)^{-1} = \left(2 + (y-2) + \frac{3}{y-2}\right)^{-1}.$$

By AM-GM we have $\left|(y-2) + \frac{3}{y-2}\right| \ge 2\sqrt{3}$. If $y - 2$ is negative, we then achieve a maximum value of $(2 - 2\sqrt{3})^{-1} < 0$, whereas if $y - 2$ is positive, we achieve a maximum value of $(2 + 2\sqrt{3})^{-1} = \boxed{\frac{\sqrt{3}-1}{4}}$.

23. **[17]** Aditya tags each point (a, b) where a and b are integers with the number $a^2 + b^2$. Andrew starts at $(0, 0)$ and every minute, he walks either up 1 unit or right 1 unit with equal probability. After 22 minutes, Megan sums the tagged numbers on each point of Andrew's walk. What is the expected value of this sum?

Proposed by Corwin Eisenbeiss and Steven Breger

Solution. Let x_t be the number Aditya tagged at Andrew's position, t minutes after Andrew began walking. Then define $y_i = x_i - x_{i-1}$ for $i = 1, \ldots, 22$. The key idea is to think about the x_i in terms of the y_i. We want to find $\mathbb{E}(x_1 + x_2 + \cdots + x_{22})$ (since $x_0 = 0$) and so rewriting this using linearity of expectation, we want

$$\mathbb{E}(x_1) + \mathbb{E}(x_2) + \cdots + \mathbb{E}(x_{22}) = \sum_{k=1}^{22} \mathbb{E}(x_k).$$

Notice that $x_0 = 0$, so

$$\begin{aligned} x_1 &= y_1 \\ x_2 &= y_1 + y_2 \\ x_3 &= y_1 + y_2 + y_3 \\ &\vdots \\ x_{22} &= y_1 + y_2 + y_3 + \cdots + y_{22} \end{aligned}$$

which means after applying linearity of expectation

$$x_k = \mathbb{E}\left(\sum_{n=1}^{k} y_n\right) = \sum_{n=1}^{k} \mathbb{E}(y_n).$$

Suppose Andrew is at (a, b) $n-1$ minutes after he begins walking. Then $x_{n-1} = a^2 + b^2$ and he has an equal probability to move to $(a+1, b)$ and $(a, b+1)$, meaning the expected value of x_n is

$$\begin{aligned} \frac{1}{2}((a+1)^2 + b^2) + \frac{1}{2}(a^2 + (b+1)^2) &= \frac{1}{2}(2a^2 + 2a + 1 + 2b^2 + 2b + 1) \\ &= a^2 + b^2 + a + b + 1. \end{aligned}$$

Furthermore, $n - 1 = a + b$ because each move Andrew either increases his x-coordinate by 1 or his y-coordinate by 1. We deduce $n = a + b + 1$ and so

$$\begin{aligned} \mathbb{E}(y_n) &= \mathbb{E}(x_n - x_{n-1}) \\ &= \mathbb{E}(x_n) - \mathbb{E}(x_{n-1}) \\ &= a^2 + b^2 + a + b + 1 - (a^2 + b^2) \\ &= a + b + 1 \\ &= n. \end{aligned}$$

Applying our formula for x_k, this means

$$x_k = \sum_{n=1}^{k} \mathbb{E}(y_n) = \sum_{n=1}^{k} n = \sum_{n=1}^{k} \binom{n}{1} = \binom{k+1}{2}$$

by the Hockey Stick identity. Then, evaluating the expected value gives

$$\sum_{k=1}^{22} \mathbb{E}(x_k) = \sum_{k=1}^{22} \binom{k+1}{2} = \sum_{k=2}^{23} \binom{k}{2} = \binom{24}{3}$$

again by the Hockey Stick identity, giving $\binom{24}{3} = \boxed{2024}$ as the answer.

24. **[17]** How many ordered pairs of positive integers (a, b) with $a > b$ and $a + b < 1000$ have an integer solution x to

$$\sqrt{a - x} + (b + x) = (a - x) - \sqrt{b + x}?$$

Proposed by Gabe Levin

Solution. We rearrange to get

$$\sqrt{a-x}+\sqrt{b+x}=a-b-2x.$$

Thus we see that

$$\sqrt{a-x}-\sqrt{b+x}=1.$$

Letting $t=\frac{a+b}{2}$ and $u=\frac{a-b}{2}-x$, we have $\sqrt{t+u}+\sqrt{t-u}=2u$ and $\sqrt{t+u}-\sqrt{t-u}=1$. Thus $\sqrt{t+u}=(2u+1)/2$, so that

$$t+u=u^2+u+\frac{1}{4},$$

and thus

$$\frac{1}{2}(a-b-2x)=u=\sqrt{t-\frac{1}{4}}=\frac{1}{2}\sqrt{2a+2b-1}.$$

Thus

$$x=\frac{1}{2}\left(a-b-\sqrt{2a+2b-1}\right).$$

Thus we just need $a>b$, $a-b$ odd, and $2a+2b-1$ a square. Let $2a+2b-1=(2k-1)^2$, so that $a+b=2k^2-2k+1$ (so $a-b$ must be odd). For each such k, there are exactly k^2-k values of (a,b) with $a>b$ and $a+b=2k^2-2k+1$. Also, $a+b<1000$ implies $2a+2b-1<1999$, so the maximum such k is 22. Thus we have a final answer of

$$\sum_{k=1}^{22}k^2-k=2\binom{23}{3}=\boxed{3542}.$$

25. **[18]** Compute all ordered pairs of integers (a,b) such that

$$\binom{289}{a+b}=9^8(b^2-9)-2b+7.$$

Proposed by Alicia Li

Solution. Take the equation modulo 17. Since $289=17^2$ is a power of 17, 17 will divide the choose coefficient so long as $a+b\neq 0,289$. This would mean $17\mid 9^8(b^2-9)-2b+7$, but $9^8=3^{16}$ which is 1 modulo 17 by Fermat's Little Theorem, so we deduce $17\mid b^2-9-2b+7$ which we may rewrite as $17\mid(b-1)^2-3$ or $(b-1)^2\equiv 3 \mod 17$. One may manually check that 3 is not a *quadratic residue* modulo 17, meaning there is no x such that $x^2\equiv 3 \mod 17$. Alternatively, one may compute the Legendre symbol using the Law of Quadratic Reciprocity to find

$$\left(\frac{3}{17}\right)=\left(\frac{17}{3}\right)=\left(\frac{2}{3}\right)=-1$$

and so 3 is not a quadratic residue modulo 17. This means $(b-1)^2\equiv 3 \mod 17$ is not possible and $a+b$ must be either 0 or 289. In either case, the choose coefficient evaluates to 1 and we are left to solve $9^8(b^2-9)-2b+7=1$ or $9^8(b^2-9)-2b+6=0$. We may factor this as

$$9^8(b+3)(b-3)-2(b-3)=0$$

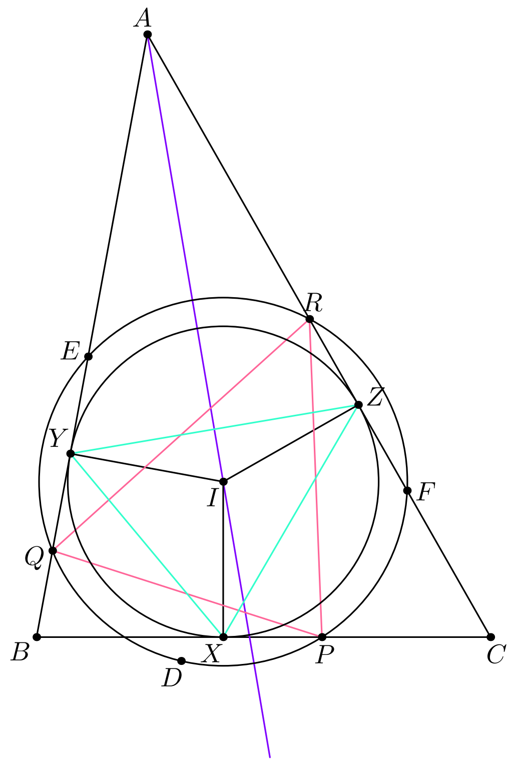
$$(9^8(b+3)-2)(b-3)=0.$$

The first term cannot evaluate to 0 (say, by taking the value modulo 9), so $b=3$ is the only possible case. Then $a+b=0$ or $a+b=289$, giving us our only solutions $\boxed{(-3,3)\text{ and }(286,3)}$.

26. **[18]** Let ABC be a triangle with $AB=7$, $AC=8$, and $BC=5$. Let P be an arbitrary point on $\overline{BC}$ and E and F be on $\overline{AB}$ and $\overline{AC}$, respectively, such that $BE=BP$ and $CF=CP$. If D is the reflection of P over the angle bisector of $\angle A$, find the minimum possible area of $\triangle DEF$.

Proposed by Corwin Eisenbeiss

Solution. Reflect E and F over the angle bisector of $\angle A$ to Q on AC and R on AB and let ω be the circumcircle of $\triangle PEF$. Let the center of ω is I and the feet from I to BC and the feet from I to BC, CA, and AB be X, Y, and Z, respectively. Shown below is a diagram.



The perpendicular bisector of $\overline{PE}$ bisects $\angle B$ and passes through I. Similarly, the perpendicular bisector of $\overline{PF}$ bisects $\angle C$ and passes through I, so we may deduce I is the incenter of $\triangle ABC$. As such, reflection over AI must preserve ω and so D, E, F, P, Q, R all lie on ω. Furthermore, $\triangle DEF \cong \triangle PQR$ by reflection so we can instead minimize the area of $\triangle PQR$.

Our main claim is $\triangle PQR \sim \triangle XYZ$. Notice $\triangle BPE \sim \triangle BXZ$ so

$$\angle XZY = \angle BXY = \angle BPE = \angle BEP = \angle PQR$$

and similarly $\triangle CPF \sim \triangle CXY$ so

$$\angle XYZ = \angle CXZ = \angle CPF = \angle CFP = 180^\circ - \angle PFQ = \angle PQR$$

and we have $\triangle PQR \sim \triangle XYZ$ as desired.

Then since all such $\triangle PQR$ are similar and have circumcenter I, the area is minimized when the circumradius is minimized, i.e. ω is the incircle and $P = X$ meaning $Q = Y$ and $R = Z$. Now, using $AY = AZ$, $BX = BY$, and $CX = CZ$, one may derive $AY = AZ = s - a$, $BX = BY = s - b$, $CX = CZ = s - c$ where s is the semiperimeter and $a = BC$, $b = CA$, $c = AB$. Then $s = \frac{5+8+7}{2} = 10$ and $a = 5$, $b = 8$, $c = 7$ so

$AY = AZ = 5$, $BX = BY = 2$, $CX = CZ = 3$. From here, we compute the area of $\triangle XYZ$ as follows

$$\begin{aligned}
[XYZ] &= [ABC] \cdot \frac{[XYZ]}{[ABC]} \\
&= [ABC]\left(\frac{[ABC]}{[ABC]} - \frac{[AYZ]}{[ABC]} - \frac{[BXY]}{[ABC]} - \frac{[CXZ]}{[ABC]}\right) \\
&= [ABC]\left(1 - \frac{\frac{1}{2}\cdot AY \cdot AZ \cdot \sin\angle A}{\frac{1}{2}\cdot AB \cdot AC \cdot \sin\angle A} - \frac{\frac{1}{2}\cdot BX \cdot BY \cdot \sin\angle B}{\frac{1}{2}\cdot BC \cdot BA \cdot \sin\angle B} - \frac{\frac{1}{2}\cdot AY \cdot AZ \cdot \sin\angle C}{\frac{1}{2}\cdot CA \cdot CB \cdot \sin\angle C}\right) \\
&= [ABC]\left(1 - \frac{AY\cdot AZ}{AB \cdot AC} - \frac{BX \cdot BY}{BC \cdot BA} - \frac{AY \cdot AZ}{CA \cdot CB}\right) \\
&= [ABC]\left(1 - \frac{5\cdot 5}{7\cdot 8} - \frac{2\cdot 2}{5 \cdot 7} - \frac{3\cdot 3}{8\cdot 5}\right) \\
&= [ABC]\left(\frac{56}{56} - \frac{25}{56} - \frac{4}{35} - \frac{9}{40}\right) \\
&= [ABC]\left(\frac{56}{56} - \frac{25}{56} - \frac{19}{56}\right) \\
&= \frac{3}{14}[ABC].
\end{aligned}$$

Using Heron's formula,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{10 \cdot 5 \cdot 2 \cdot 3} = 10\sqrt{3}$$

and so the answer is $\frac{3}{14} \cdot 10\sqrt{3} = \boxed{\frac{15\sqrt{3}}{7}}$.

27. **[19]** Consider a 40-cycle graph C and let A be a vertex of C, let B be a neighbor of A, and let A' and B' be their antipodes, respectively. Consider the graph C', whose vertices are exactly the vertices of C, and whose edges are the edges of C, as well as the edges connecting each pair of antipodal vertices, for a total of 60 edges. Let a *borisaurus* be a partition of a graph's vertex set into 4-vertex subsets such that in each subset S of the partition, the following two conditions hold:

   (i) for any s in S, there exists s' such that s and s' are connected by an edge,

   (ii) for any s, s' in S with s and s' connected by an edge, there is a third vertex s'' in S such that s'' is connected by an edge to at least one of s and s'.

   How many borisauruses on C' satisfy the additional condition that if both A and B or both A' and B' are in the same subset S of the borisaurus, then that set is precisely $\{A, B, A', B'\}$?

   Proposed by Gabe Levin

   Solution. Good luck!

28. **[19]** Let the multivariable polynomial $P_n(x, y)$ be defined by

$$P_n(x,y) = \sum_{i=0}^{n} x^i y^{n-i}$$

   for all n. Compute the number of ordered pairs of complex numbers (x, y) for which $P_{14}(x+1, y+4) = P_{23}(x+2, y+3) = 0$.

   Proposed by Corwin Eisenbeiss

   Solution. Notice that if we divide $P_n(x, y)$ by y^n ($y = 0$ means $x = 0$, which we will see is always a solution), we get a polynomial in terms of $\frac{x}{y}$. In particular,

$$\frac{P_n(x,y)}{y^n} = \left(\frac{x}{y}\right)^n + \left(\frac{x}{y}\right)^{n-1} + \cdots + \left(\frac{x}{y}\right) + 1,$$

meaning $P_n(x,y)=0$ really means $\frac{x}{y}=\omega$ where ω is an $n+1$th root of unity not equal to 1. Then for example $P_{14}(x+1,y+4)=0$ tells us $\frac{x+1}{y+4}=\omega$ for $\omega\neq 1$ as a 15th root of unity and $P_{23}(x+2,y+3)=0$ tells us $\frac{x+2}{y+3}=\zeta$ for $\zeta\neq 1$ as a 24th root of unity. Although these are complex numbers, the first set of equations is 14 (complex) lines through $(-1,-4)$ and the second set of equations is 23 lines through $(-2,-3)$. Intuitively, any pair of these lines should give an intersection so long as they are not "parallel" which would give an answer of $14\cdot 23$, so our main claim will be to show this intuition is largely true. Of course, this statement generalizes past these numbers, but for convenience we will use the given numbers.

Note the first equation implies $(x+1)=(y+4)\omega \implies x=y\omega+4\omega-1$ and $(x+2)=(y+3)\zeta \implies x=y\zeta+3\zeta-2$. If there is no solution, then $y\omega-4\omega-1=y\zeta+3\zeta-2$ has no solution, but rearranging this will give the solution $y=\frac{3\zeta-2+4\omega+1}{\omega-\zeta}$, so indeed, as long as $\omega\neq\zeta$, we have exactly 1 intersection. Note that $\omega=\zeta$ will happen at all numbers that are both 15th roots of unity and 24th roots of unity. There are $\gcd(15,24)=3$ of these, but 1 is the number 1 so we only need to remove 2 cases here.

Furthermore, each point is the intersection of at most 2 lines **except potentially the points** $(-1,-4)$ **and** $(-2,-3)$. This is because if three distinct lines intersect at a point, two lines share either $(-1,-4)$ or $(-2,-3)$ and that common point and are thus not distinct. Of course, if this common point is exactly one of the two, these lines actually only share one point, so we need to keep note of if $(-2,-3)$ is a solution to $\frac{x+1}{y+4}=\omega \implies \omega=-1$ or if $(-1,-4)$ is a solution to $\frac{x+2}{y+3}=\zeta \implies \zeta=-1$. Notice that ω is a 15th root of unity so $\omega=-1$ is impossible, but ζ is a 24th root of unity so $\zeta=-1$ actually **is** possible. This means this line will intersect the 14 lines at 1 point, causing us to have 13 less intersections than expected.

Summing it all up, there are $14\cdot 23-2-13=\boxed{307}$ ordered pairs.

29. **[20]** Let $ABCD$ be a cyclic quadrilateral with AD and BC meeting at T, AC and BD meeting at P, and AB and CD meeting at a 60° angle. Point $M\neq P$ is the intersection of the circumcircles of $\triangle PAB$ and $\triangle PCD$ and V is the reflection of M over CD. Given that $\frac{AB}{CD}=\frac{3}{5}$ and $MV=224$, compute TV.

Proposed by Corwin Eisenbeiss

Solution. Good luck!

30. **[20]** For primes p, let $f(p)$ be the number of ordered pairs (a,b) of integers such that $0\leq a,b<p$ and p divides $a^5-10a^3b^2+10a^2b^3-b^5-1$ and $5a^4b-10a^3b^2+5ab^4-b^5$. Let S_n be the set of the first n primes. Compute

$$\lim_{n\to\infty}\frac{1}{n}\sum_{p\in S_n} f(p)^2.$$

Proposed by Tanvir Ahmed

Solution. Let ω be a primitive 3rd root of unity, so $\omega^2+\omega+1=0$, which means $\omega^2=-1-\omega$. Also, let $P(a,b)=a^5-10a^3b^2+10a^2b^3-b^5$ and $Q(a,b)=5a^4b-10a^3b+5ab^4-b^5$. The main observation is the following:

$$\begin{aligned}(a+b\omega)^5 &= a^5+5a^4b\omega+10a^3b^2\omega^2+10a^2b^3+5ab^4\omega+b^5\omega^2\\ &= (a^5-10a^3b^2+10a^2b^3-b^5)+(5a^4b-10a^3b+5ab^4-b^5)\omega\\ &= P(a,b)+Q(a,b)\omega,\end{aligned}$$

and we see the polynomials given in the problem pop out. Also notice that for all the pairs that are counted by $f(p)$, $a,b\in\{0,1,\ldots,p-1\}$, and this set covers all residues mod p exactly once. Thus, we can think about $f(p)$ as counting pairs in $\mathbb{F}_p^2$. Also notice that the conditions given for pairs counted by $f(p)$ are just $P(a,b)\equiv 1 \pmod{p}$ and $Q(a,b)\equiv 0 \pmod{p}$.

For the first case, suppose that $\omega\in\mathbb{F}_p$, meaning there is a primitive 3rd root of unity mod p. This is equivalent to $p\equiv 1 \pmod{3}$. In this case, $(a+b\omega)^5=1$ in $\mathbb{F}_p$. However, ω^2 is also a primitive 3rd root of unity, so by the same reasoning, we can arrive at $(a+b\omega^2)^5=1$. If $p\equiv 1 \pmod{5}$, then there are 5 5th roots of unity mod p, so there are 5 choices for $a+b\omega$ and 5 for $a+b\omega^2$. For each of the 25 combinations of choices, we can solve a system of equations to get unique values for a and b. If we have a pair (a,b) such that $a+b\omega$ and $a+b\omega^2$ are 5th roots, we can also expand out their 5th powers to get $P(a,b)+Q(a,b)\omega \equiv 0 \pmod p$ and $P(a,b)+Q(a,b)\omega^2 \equiv 1 \pmod p$. Taking suitable linear combinations will produce $P(a,b)\equiv 1 \pmod p$ and $Q(a,b)\equiv 0 \pmod p$. In other words, if $p\equiv 1 \pmod 3$ and $p \equiv 1 \pmod 5$, $f(p)=25$.

In the case where $p\equiv 1 \pmod 3$ and $p \not\equiv 1 \pmod 5$, we still need $a+b\omega$ and $a+b\omega^2$ (both elements of $\mathbb{F}_p$) to be 5th roots, but 1 is the only 5th root. Thus, in this case $f(p)=1$, corresponding to the ordered pair $(a,b)=(1,0)$.

In the case where $p\equiv 2 \pmod 3$, $\omega \notin \mathbb{F}_p$, so we consider the field extension $\mathbb{F}_p[\omega]=\mathbb{F}_{p^2}$. Once again, we need to count the solutions to $(a+b\omega)^5=1$ or $a+b\omega$ being a 5th root of unity where $a,b\in\mathbb{F}_p$. However, $\mathbb{F}_p[\omega]$ is a vector space over $\mathbb{F}_p$ with basis $\{1,\omega\}$. Therefore, if we have $a+b\omega=\zeta$ for some 5th root $\zeta\in\mathbb{F}_{p^2}$, there is exactly one solution for (a,b). Therefore, in this case, $f(p)$ is just the number of 5th roots of unity in $\mathbb{F}_{p^2}$. Since $\mathbb{F}_{p^2}^\times$, the multiplicative group of $\mathbb{F}_{p^2}$, is cyclic of order p^2-1, there are 5 5th roots of unity if $5 \mid p^2-1$ and 1 root otherwise. Thus, $f(p)=5$ if $p\equiv 1,4 \pmod 5$ and $f(p)=1$ if $p\equiv 2,3 \pmod 5$.

Now we summarize our findings. For primes $p>5$, the value of $f(p)$ only depends on p mod 15. We can use the Chinese Remainder Theorem and all the cases we got earlier to write the following:

$$f(p)=\begin{cases} 25 & p\equiv 1 \pmod{15} \\ 5 & p\equiv 11,14 \pmod{15} \\ 1 & p\equiv 2,4,7,8,13 \pmod{15} \end{cases}.$$

For the final computation, for integers a with $\gcd(a,15)=1$, let $\pi_a(n)$ be the number of elements of S_n (which are primes) that are congruent to a mod 15. By Dirichlet's Theorem, we have

$$\lim_{n\to\infty}\frac{\pi_a(n)}{|S_n|}=\lim_{n\to\infty}\frac{\pi_a(n)}{n}=\frac{1}{\varphi(15)}=\frac{1}{8},$$

or in other words, $\pi_a(n)\sim\frac{n}{8}$. Thus, we have

$$\begin{aligned}\frac{1}{n}\sum_{p\in S_n} f(p)^2 &\sim \frac{1}{n}\cdot\frac{n}{8}(25^2+5^2+5^2+1^2+1^2+1^2+1^2+1^2) \\ &= \frac{680}{8} \\ &= 85\end{aligned}$$

where we obtained this by summing over primes congruent to each of the coprime residues mod 15. The above calculation is just another way of saying

$$\lim_{n\to\infty}\frac{1}{n}\sum_{p\in S_N} f(p)^2=\boxed{85}.$$

Solution. This solution is functionally the same as the previous, but is phrased more directly in terms of algebraic number theory. We use the same notation introduced in the previous solution and start from $(a+b\omega)^5=P(a,b)+Q(a,b)\omega$. In this solution, the ring $\mathbb{Z}[\omega]$ will play a crucial role.

If (a,b) is a pair counted by $f(p)$, we know $P(a,b)\equiv 1 \pmod p$ and $Q(a,b)\equiv 1 \pmod p$. This means that $(a+b\omega)^5\equiv 1 \pmod p$, with respect to the ring $\mathbb{Z}[\omega]$. In other words, in the quotient $\mathbb{Z}[\omega]/(p)$, $(a+b\omega)^5=1$, where (p) denotes the ideal generated by p. Now we have some cases.

If $p\equiv 1 \pmod 3$, p is not prime in $\mathbb{Z}[\omega]$, and there exist primes $\pi_1,\pi_2\in\mathbb{Z}[\omega]$ such that $p=\pi_1\pi_2$ and $N(\pi_1)=N(\pi_2)=p$. Thus, $\mathbb{Z}[\omega]/(p)\cong\mathbb{Z}[\omega]/(\pi_1)\times\mathbb{Z}[\omega]/(\pi_2)\cong\mathbb{F}_p^2$ by the Chinese Remainder Theorem. Now we need to find the number of elements of $\mathbb{F}_p^2$ whose 5th power is 1. Such an element can only come about by combining two elements of $\mathbb{F}_p$ whose 5th power is 1. If $p\equiv 1 \pmod 5$, there are 5 5th roots of unity in $\mathbb{F}_p$, so there are 25 solutions to $x^5 = 1$ in $\mathbb{F}_p^2$, meaning $f(p) = 25$ for this case. If $p \equiv 1 \pmod 5$ though, there is only 1 5th root of unity in $\mathbb{F}_p$ and consequently only one solution in $\mathbb{F}_p^2$ as well. Thus, $f(p) = 1$ for $p \equiv 1 \pmod 3$ and $p \not\equiv 1 \pmod 5$.

If $p \equiv 2 \pmod 3$, then p is prime in $\mathbb{Z}[\omega]$ and $\mathbb{Z}[\omega]/(p)$ forms a field of order $N(p) = p^2$, which is just $\mathbb{F}_{p^2}$. There are 5 primitive 5th roots of unity here if and only if $5 \mid p^2 - 1$, so $f(p) = 5$ if $p \equiv 2 \pmod 3$ and $p \equiv 1, 4 \pmod 5$. Otherwise, if $p \equiv 2 \pmod 3$ and $p \equiv 2, 3 \pmod 5$, then $f(p) = 1$. The computation step with Dirichlet's Theorem proceeds as in the previous solution.