NYCMT 2025-2026 Homework #6

NYCMT

January 30, 2025 - February 6, 2026

These problems are due February 6th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve.

Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on February 6th, you can scan your solutions and email them to captains@nycmathteam.org. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or use one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

Problem 1. Find all positive integers n such that $2^n - 1$ is not divisible by any primes greater than 7.

Problem 2. Consider a directed graph on n vertices with no cycles. Prove that one can label the edges of the graph with positive integers less than n such that the sum of the labels of edges on any two paths from A to B is the same.

Here, a *cycle* is a series of vertices $v_1, \ldots, v_n$ such that there is a directed edge pointing from v_i to v_{i+1} for every $1 \le i \le n$, with indices taken modulo n.

Problem 3. Evaluate the sum

$$\sum_{i=0}^{1000} \frac{\binom{1000}{i}}{\binom{2026}{i}}.$$

Problem 4. The distinct roots of the cubic $x^3 - ax^2 + 6x + 7$ are p, q, and r. If $(p+q)(q+r)(r+p) = a$, find a.

Problem 5. Suppose that $\triangle ABC$ has $AB = 5$, $AC = 6$, and $BC = 7$. Let M be the midpoint of segment BC and let P be a point inside $\triangle ABC$ such that $\angle BPM = \angle B$ and $\angle CPM = \angle C$. Find the length of segment AP.

Problem 6. [Bonus ⧫]: The writers of the Fall NYCTC were very sad that no one solved these problems. Please make them happy by solving one of the two.

27. Consider the 60-edge graph C consisting of a 40-cycle and edges between antipodal vertices. Let A and B be neighboring vertices (not antipodes) and let A' and B' be their respective antipodes. Let a *borisaurus* be a partition of a graph's vertex set into connected 4-element subsets (i.e. the maximal subgraph containing the four vertices is connected). How many borisauruses on C satisfy the additional condition that if both A and B or both A' and B' are in the same subset S of the borisaurus, then that set is precisely $\{A, B, A', B'\}$?

29. Let $ABCD$ be a cyclic quadrilateral with AD and BC meeting at T, AC and BD meeting at P, and AB and CD meeting at a $60°$ angle. Point $M \neq P$ is the intersection of the circumcircles of $\triangle PAB$ and $\triangle PCD$ and V is the reflection of M over CD. Given that $\frac{AB}{CD} = \frac{3}{5}$ and $MV = 192$, compute TV.