HMMT November 2022

November 12, 2022

Team Round

1. **[20]** Two linear functions $f(x)$ and $g(x)$ satisfy the properties that for all x,

   - $f(x) + g(x) = 2$
   - $f(f(x)) = g(g(x))$

   and $f(0) = 2022$. Compute $f(1)$.

   Proposed by: Vidur Jasuja

   Answer: $\boxed{1}$

   Solution 1: Firstly, $f(x)$ and $g(x)$ must intersect – otherwise, $f(x) = g(x) = 1$, which can't be true.

   Secondly, suppose they intersect at a, so that $f(a) = g(a) = c$. Then $f(c) = g(c) \implies f(c) = g(c) = 1$. But then, $a = c$, and $c = 1$. So $f(1) = 1$, and we're done.

   Solution 2: We will solve the problem manually, setting $f(x) = ax + 2022$ and $g(x) = -ax - 2020$. Then $f(f(x)) = a^2x + 2022a + 2022$, while $g(g(x)) = a^2x + 2020a - 2020$.

   Then, $a = -2021$, so then $f(1) = 1$.

2. **[25]** What is the smallest r such that three disks of radius r can completely cover up a unit disk?

   Proposed by: Gabriel Wu

   Answer: $\boxed{\frac{\sqrt{3}}{2}}$

   Solution: Look at the circumference of the unit disk. Each of the disks must be capable of covering up at least $\frac{1}{3}$ of the circumference, which means it must be able to cover a chord of length $\sqrt{3}$. Thus, $\frac{\sqrt{3}}{2}$ is a lower bound for r. This bound is achievable: place the three centers of the disks symmetrically at a distance of $\frac{1}{2}$ from the center of the unit disk.

3. **[30]** Find the number of ordered pairs (A, B) such that the following conditions hold:

   - A and B are disjoint subsets of $\{1, 2, \ldots, 50\}$.
   - $|A| = |B| = 25$
   - The median of B is 1 more than the median of A.

   Proposed by: Papon Lapate

   Answer: $\boxed{\binom{24}{12}^2}$

   Solution: The median of both sets, which we will call a and b respectively, are more than exactly 12 of the members in their own set. Since a and b are consecutive, they must also be higher than the lower half of the other set and lower than the higher half of the other set, meaning that they are both higher than exactly 24 numbers in $\{1, 2, ..., 50\} - \{a, b\}$. Thus, $a = 25$ and $b = 26$.

   The 24 lower numbers can be divided into the two groups (with 12 in each group) in $\binom{24}{12}$ ways. Similarly, the 24 higher numbers can be divided into the two groups in $\binom{24}{12}$ ways. Thus, the answer is $\binom{24}{12}^2$.

4. **[35]** You start with a single piece of chalk of length 1. Every second, you choose a piece of chalk that you have uniformly at random and break it in half. You continue this until you have 8 pieces of chalk. What is the probability that they all have length $\frac{1}{8}$?

Proposed by: Evan Erickson, Gabriel Wu

Answer: $\boxed{\frac{1}{63}}$

Solution 1: There are 7! total ways to break the chalks. How many of these result in all having length $\frac{1}{8}$? The first move gives you no choice. Then, among the remaining 6 moves, you must apply 3 breaks on the left side and 3 breaks on the right side, so there are $\binom{6}{3} = 20$ ways to order those. On each side, you can either break the left side or the right side first. So the final answer is

$$\frac{20 \cdot 2^2}{7!} = \frac{1}{63}.$$

Solution 2: We know there are 7! ways to break the chalk in total.

Now, if we break up the chalk into 8 pieces, we can visualize the breaks as a binary decision tree. Each round we select a node and break that corresponding piece of chalk, expanding it into two branch nodes. The final tree of our desired configuration will have three layers.

We can figure out how many different ordering we can do this in with recursion. If b_n is the number of ways to expand a binary tree with n layers, we have $b_1 = 1$. Now when we expand a node with $k+1$ layers, we will expand either the k-layered tree on the left or right, these moves can be ordered in $\binom{2^{k+1}-2}{2^k-1}$ ways. For each one of these trees, there are b_k ways to decide these moves. So we have $b_{k+1} = \binom{2^{k+1}-2}{2^k-1} b_k^2$. So $b_2 = \binom{2}{1} \cdot 1^2 = 2$, $b_3 = \binom{6}{3} \cdot 2^2 = 20 \cdot 2^2$. Thus, the final answer is

$$\frac{20 \cdot 2^2}{7!} = \frac{1}{63}.$$

5. **[40]** A triple of positive integers (a, b, c) is *tasty* if $\operatorname{lcm}(a, b, c) \mid a + b + c - 1$ and $a < b < c$. Find the sum of $a + b + c$ across all tasty triples.

Proposed by: Isaac Zhu

Answer: $\boxed{44}$

Solution: The condition implies $c \mid b + a - 1$. WLOG assume $c > b > a$; since $b + a - 1 < 2c$ we must have $b + a - 1 = c$. Substituting into $b \mid a + c - 1$ and $a \mid c + b - 1$ gives

$$b \mid 2a - 2$$

$$a \mid 2b - 2.$$

Since $2a - 2 < 2b$ we must either have $a = 1$ (implying $a = b$, bad) or $2a - 2 = b \implies a \mid 4a - 6 \implies a = 2, 3, 6$. If $a = 2$ then $b = 2$. Otherwise, if $a = 3$ we get $(3, 4, 6)$ and if $a = 6$ we get $(6, 10, 15)$, so answer is $13 + 31 = 44$.

6. **[45]** A triangle XYZ and a circle ω of radius 2 are given in a plane, such that ω intersects segment $\overline{XY}$ at the points A, B, segment $\overline{YZ}$ at the points C, D, and segment $\overline{ZX}$ at the points E, F. Suppose that $XB > XA$, $YD > YC$, and $ZF > ZE$. In addition, $XA = 1, YC = 2, ZE = 3$, and $AB = CD = EF$. Compute AB.

Proposed by: Rishabh Das

Answer: $\boxed{\sqrt{10}-1}$

Solution: Let $d = AB$ and $x = d/2$ for ease of notation. Let the center of $(ABCDEF)$ be I. Because $AB = CD = EF$, the distance from I to AB, CD, and EF are the same, so I is the incenter of $\triangle XYZ$. Let $\triangle XYZ$ have inradius r.

By symmetry, we have $XF = 1$, $YB = 2$, and $ZD = 3$. Thus, $\triangle XYZ$ has side lengths $d+3$, $d+4$, and $d+5$. Heron's Formula gives the area of $\triangle XYZ$ is

$$K = \sqrt{(3x+6)(x+2)(x+1)(x+3)} = (x+2)\sqrt{3(x+1)(x+3)},$$

while $K = rs$ gives the area of $\triangle XYZ$ as

$$K = (3x+6)\,r.$$

Equating our two expressions for K, we have

$$(x+2)\sqrt{3(x+1)(x+3)} = (3x+6)\,r \implies \sqrt{3(x+1)(x+3)} = 3r \implies (x+1)(x+3) = 3r^2.$$

The Pythagorean Theorem gives $x^2 + r^2 = 4$, so $r^2 = 4 - x^2$. Plugging this in and expanding gives

$$(x+1)(x+3) = 3\left(4 - x^2\right) \implies 4x^2 + 4x - 9 = 0.$$

This has roots $x = \frac{-1\pm\sqrt{10}}{2}$, and because $x > 0$, we conclude that $d = \sqrt{10} - 1$.

7. **[45]** Compute the number of ordered pairs of positive integers (a, b) satisfying the equation

$$\gcd(a, b) \cdot a + b^2 = 10000.$$

Proposed by: Vidur Jasuja

Answer: $\boxed{99}$

Solution 1: Let $\gcd(a, b) = d, a = da', b = db'$. Then, $d^2(a' + b'^2) = 100^2$. Consider each divisor d of 100. Then, we need to find the number of solutions in coprime integers to $a' + b'^2 = \frac{100^2}{d^2}$. Note that every $b' < 100/d$ coprime to $\frac{100^2}{d^2}$ satisfies this equation, which is equivalent to being coprime to $\frac{100}{d}$, so then there are $\varphi\left(\frac{100}{d}\right)$ choices for each d, except for $d = 100$, which would count the solution $(0, 100)$. Then we just need $\sum_{d|n} \varphi\left(\frac{100}{d}\right) - 1 = 100 - 1 = 99$.

Solution 2: Note that b must be at most 99 in order for a to be positive. Now we claim that each choice of $b \in \{1, 2, \ldots, 99\}$ corresponds to exactly one value of a that satisfies the equation.

To see why this is true, we rewrite the formula as

$$\gcd(a, b) \cdot a = (100 - b)(100 + b).$$

For any prime p, let $v_p(n)$ be the largest integer k such that $p^k | n$. We wish to show that once we fix b, the value $v_p(a)$ is uniquely determined for all p (which will give us a unique solution for a). Applying v_p to both sides of the equation gives us

$$\min(v_p(b), v_p(a)) + v_p(a) = v_p(100 - b) + v_p(100 + b).$$

We immediately see that there is at most one solution for $v_p(a)$ since the left-hand side increases with $v_p(a)$. Further, as we increment $v_p(a)$, the left-hand side takes on all even numbers up to $2v_p(b)$, and then all integers larger than $2v_p(b)$. So to show that a solution exists, we need to prove that the right-hand side is always either even or at least $2v_p(b)$.

If $v_p(b) \neq v_p(100)$, then $v_p(100 - b) = v_p(100 + b) = \min(v_p(100), v_p(b))$. In this case, $v_p(100 - b) + v_p(100 + b)$ is clearly even. Otherwise, assume $v_p(b) = v_p(100)$. Then $v_p(100 - b) \geq v_p(b)$ and $v_p(100 + b) \geq v_p(b)$, so $v_p(100 - b) + v_p(100 + b) \geq 2v_p(b)$ as desired. Thus, there is always a unique solution for $v_p(a)$. Once we fix $1 \leq b \leq 99$, the value of a is uniquely determined, so the answer is 99.

8. **[50]** Consider parallelogram $ABCD$ with $AB > BC$. Point E on $\overline{AB}$ and point F on $\overline{CD}$ are marked such that there exists a circle ω_1 passing through A, D, E, F and a circle ω_2 passing through B, C, E, F. If ω_1, ω_2 partition $\overline{BD}$ into segments $\overline{BX}, \overline{XY}, \overline{YD}$ in that order, with lengths $200, 9, 80$, respectively, compute BC.

Proposed by: Albert Wang

Answer: $\boxed{51}$

Solution: We want to find $AD = BC = EF$. So, let EF intersect BD at O. It is clear that $\triangle BOE \sim \triangle DOF$. However, we can show by angle chase that $\triangle BXE \sim \triangle DYF$:

$$\angle BEG = \angle ADG = \angle CBH = \angle DFH.$$

This means that $\overline{EF}$ partitions $\overline{BD}$ and $\overline{XY}$ into the same proportions, i.e. 200 to 80. Now, let $a = 200, b = 80, c = 9$ to make computation simpler. O is on the radical axis of ω_1, ω_2 and its power respect to the two circles can be found to be

$$\left(a + \frac{ac}{a+b}\right)\frac{bc}{a+b} = \frac{abc(a+b+c)}{(a+b)^2}$$

However, there is now x for which $OE = ax, OF = bx$ by similarity. This means $x^2 = \frac{c(a+b+c)}{(a+b)^2}$. Notably, we want to find $(a+b)x$, which is just

$$\sqrt{c(a+b+c)} = \sqrt{9 \cdot 289} = 51.$$

9. **[50]** Call an ordered pair (a, b) of positive integers *fantastic* if and only if $a, b \leq 10^4$ and

$$\gcd(a \cdot n! - 1, a \cdot (n+1)! + b) > 1$$

for infinitely many positive integers n. Find the sum of $a + b$ across all fantastic pairs (a, b).

Proposed by: Pitchayut Saengrungkongka

Answer: $\boxed{5183}$

Solution: We first prove the following lemma, which will be useful later.

Lemma: Let p be a prime and $1 \leq n \leq p-1$ be an integer. Then, $n!(p-1-n)! \equiv (-1)^{n-1} \pmod{p}$.

Proof. Write

$$\begin{aligned} n!(p-n-1)! &= (1 \cdot 2 \cdots n)((p-n-1) \cdots 2 \cdot 1) \\ &\equiv (-1)^{p-n-1}(1 \cdot 2 \cdots n)((n+1) \cdots (p-2)(p-1)) \pmod{p} \\ &= (-1)^n (p-1)! \\ &\equiv (-1)^{n-1} \pmod{p} \end{aligned}$$

(where we have used Wilson's theorem). This implies the result. □

Now, we begin the solution. Suppose that a prime p divides both $a \cdot n! - 1$ and $a \cdot (n+1)! + b$. Then, since

$$-b \equiv a \cdot (n+1)! \equiv (n+1) \cdot (a \cdot n!) \equiv (n+1) \pmod{p},$$

we get that $p \mid n + b + 1$. Since we must have $n < p$ (or else $p \mid n!$), we get that, for large enough n, $n = p - b - 1$. However, by the lemma,

$$a(-1)^{b-1} \equiv a \cdot b!(p-1-b)! = a \cdot b!n! \equiv b! \pmod{p}.$$

This must hold for infinitely many p, so $a = (-1)^{b-1}b!$. This forces all fantastic pairs to be in form $((2k-1)!, 2k-1)$.

Now, we prove that these pairs all work. Take $n = p - 2k$ for all large primes p. Then, we have

$$\begin{aligned} a \cdot n! &\equiv (2k-1)!(p-2k)! \\ &\equiv (-1)^{2k} \equiv 1 \pmod p \\ a \cdot (n+1)! &\equiv (n+1) \cdot (a \cdot n!) \\ &\equiv (p-2k+1) \cdot 1 \equiv -(2k-1) \pmod p, \end{aligned}$$

so p divides the gcd.

The answer is $(1+1) + (6+3) + (120+5) + (5040+7) = 5183$.

10. **[60]** There is a unit circle that starts out painted white. Every second, you choose uniformly at random an arc of arclength 1 of the circle and paint it a new color. You use a new color each time, and new paint covers up old paint. Let c_n be the expected number of colors visible after n seconds. Compute $\lim_{n\to\infty} c_n$.

Proposed by: Gabriel Wu

Answer: $\boxed{4\pi}$

Solution 1: Notice that colors always appear in contiguous arcs on the circle (i.e. there's never a color that appears in two disconnected arcs). So the number of distinct visible colors is equal to the number of radii that serve as boundaries between colors. Each time we place a new color, we create 2 more of these radii, but all of the previous radii have a $p = \frac{1}{2\pi}$ chance of being covered up.

It is well-known that, given an event that occurs with probability p, it takes an expected $\frac{1}{p}$ trials for it to happen – this means that any given radii has an expected lifespan of 2π before it gets covered up (i.e. it remains visible for 2π turns on average). Thus, over the course of $n \gg 1$ turns there are $2n$ total radii created, each lasting an average of 2π turns, so there are $\frac{4\pi n}{n} = 4\pi$ radii visible on average each turn.

A more rigorous way to see this is the two radii created on the most recent turn have a probability 1 of being exposed; the two radii created last turn each have a probability $1-p$ of being exposed; the two radii created two turns ago each have a probability $(1-p)^2$ of being exposed, and so on. Thus, on turn n, the expected number of exposed radii is

$$2(1 + (1-p) + (1-p)^2 + \cdots + (1-p)^{n-1}).$$

This geometric series converges to $\frac{2}{p}$ as n grows.

Solution 2: Notice that colors always appear in contiguous arcs on the circle (i.e. there's never a color that appears in two disconnected arcs). So the number of distinct visible colors is equal to the number of radii that serve as boundaries between colors. Each time we place a new color, we create 2 more of these radii. However, if the expected number of colors after turn n is constant, we must expect to remove 2 of these radii each turn. This is only possible when there are 4π total radii, since we expect to remove $\frac{1}{2\pi}$ of them each time.

Solution 3: Consider the probability that the k-th last added arc is visible. Suppose there are j arcs after the k-th last arc that partially covers this arc. Then the probability that the k-th last arc is still visible is $\frac{j+1}{2^j}$, since this is equivalent to randomly choosing j positions within the k-th last arc to place an arc in, then randomly choosing a direction, and there are 2^j ways to choose directions and $j+1$ of them are good. The probability that any arc partially covers the k-th last arc is $\frac{2}{2\pi}$. Putting everything together, the probability that the k-th last arc is visible is

$$\sum_{j=0}^{k-1} \frac{j+1}{2^j} \cdot \binom{k-1}{j} \cdot \left(\frac{2}{2\pi}\right)^j \cdot \left(1 - \frac{2}{2\pi}\right)^{k-1-j}$$

so the answer is

$$1+\sum_{k=2}^{n}\sum_{j=0}^{k-1}\frac{j+1}{2^j}\cdot\binom{k-1}{j}\cdot\left(\frac{2}{2\pi}\right)^j\cdot\left(1-\frac{2}{2\pi}\right)^{k-1-j}$$

(as the last arc is definitely visible).

We can write this as

$$1+\sum_{k=2}^{n}\sum_{j=0}^{k-1}(j+1)\cdot\binom{k-1}{j}\cdot\left(\frac{1}{2\pi}\right)^j\cdot\left(1-\frac{2}{2\pi}\right)^{k-1-j}.$$

Now,

$$\sum_{j=0}^{k-1}1\cdot\binom{k-1}{j}\cdot\left(\frac{1}{2\pi}\right)^j\cdot\left(1-\frac{2}{2\pi}\right)^{k-1-j}=\left(\frac{1}{2\pi}+1-\frac{2}{2\pi}\right)^{k-1}$$

by binomial theorem. We can write $j\cdot\binom{k-1}{j}=(k-1)\binom{k-2}{j-1}$, so

$$\begin{aligned}\sum_{j=0}^{k-1}j\cdot\binom{k-1}{j}\cdot\left(\frac{1}{2\pi}\right)^j\cdot\left(1-\frac{2}{2\pi}\right)^{k-1-j}&=(k-1)\sum_{j=1}^{k-1}\binom{k-2}{j-1}\cdot\left(\frac{1}{2\pi}\right)^j\cdot\left(1-\frac{2}{2\pi}\right)^{k-1-j}\\&=(k-1)\sum_{j=0}^{k-2}\binom{k-2}{j}\cdot\left(\frac{1}{2\pi}\right)^{j+1}\cdot\left(1-\frac{2}{2\pi}\right)^{k-2-j}\\&=\frac{k-1}{2\pi}\cdot\left(\frac{1}{2\pi}+1-\frac{2}{2\pi}\right)^{k-2}.\end{aligned}$$

Therefore the answer is

$$1+\sum_{k=2}^{n}\left(1-\frac{1}{2\pi}+\frac{k-1}{2\pi}\right)\left(1-\frac{1}{2\pi}\right)^{k-2}$$

which is an arithmetic sequence times a geometric sequence. Standard techniques simplify this to 4π.

Solution 4: We solve for the lifespan of an arc. Let $f(x)$ represent the expected number of turns an arc of length $2\pi x$ will remain visible. Our final answer will be to calculate $f(\frac{1}{2\pi})$ Then we get the recurrence

$$f(x)=1+(\frac{2\pi-1}{2\pi}-x)f(x)+2\int_0^x f(x)\,dx$$

The 1 term comes from counting the fact that the arc is visible during the current turn. The $(\frac{2\pi-1}{2\pi}-x)f(x)$ term comes from the fact that there is a $\frac{2\pi-1}{2\pi}-x$ chance that the next arc will not intersect the current arc, in which case the current arc would get an extra $f(x)$ turns to live. The integral comes from the fact that we want to take the average of $f(y)$ for $y\sim_r[0,x]$, which corresponds to the next arc covering up $2\pi(x-y)$ of the current one.

If we differentiate both sides, we end up with a differential equation. Solve it via separation. We end up with $f(x)=4\pi^2x+2\pi$. Plugging in $x=\frac{1}{2\pi}$ gets us 4π.

In general, the answer is $f(x)=\frac{x}{k^2}+\frac{1}{k}$, where we are placing arcs of length $2\pi k$.