HMMT November
November 12, 2022

Team Round

1. **[20]** Two linear functions $f(x)$ and $g(x)$ satisfy the properties that for all x,

   - $f(x) + g(x) = 2$
   - $f(f(x)) = g(g(x))$

   and $f(0) = 2022$. Compute $f(1)$.

2. **[25]** What is the smallest r such that three disks of radius r can completely cover up a unit disk?

3. **[30]** Find the number of ordered pairs (A, B) such that the following conditions hold:

   - A and B are disjoint subsets of $\{1, 2, \ldots, 50\}$.
   - $|A| = |B| = 25$
   - The median of B is 1 more than the median of A.

4. **[35]** You start with a single piece of chalk of length 1. Every second, you choose a piece of chalk that you have uniformly at random and break it in half. You continue this until you have 8 pieces of chalk. What is the probability that they all have length $\frac{1}{8}$?

5. **[40]** A triple of positive integers (a, b, c) is *tasty* if $\operatorname{lcm}(a, b, c) \mid a + b + c - 1$ and $a < b < c$. Find the sum of $a + b + c$ across all tasty triples.

6. **[45]** A triangle XYZ and a circle ω of radius 2 are given in a plane, such that ω intersects segment $\overline{XY}$ at the points A, B, segment $\overline{YZ}$ at the points C, D, and segment $\overline{ZX}$ at the points E, F. Suppose that $XB > XA$, $YD > YC$, and $ZF > ZE$. In addition, $XA = 1, YC = 2, ZE = 3$, and $AB = CD = EF$. Compute AB.

7. **[45]** Compute the number of ordered pairs of positive integers (a, b) satisfying the equation

$$\gcd(a, b) \cdot a + b^2 = 10000.$$

8. **[50]** Consider parallelogram $ABCD$ with $AB > BC$. Point E on $\overline{AB}$ and point F on $\overline{CD}$ are marked such that there exists a circle ω_1 passing through A, D, E, F and a circle ω_2 passing through B, C, E, F. If ω_1, ω_2 partition $\overline{BD}$ into segments $\overline{BX}, \overline{XY}, \overline{YD}$ in that order, with lengths $200, 9, 80$, respectively, compute BC.

9. **[50]** Call an ordered pair (a, b) of positive integers *fantastic* if and only if $a, b \leq 10^4$ and

$$\gcd(a \cdot n! - 1, a \cdot (n+1)! + b) > 1$$

   for infinitely many positive integers n. Find the sum of $a + b$ across all fantastic pairs (a, b).

10. **[60]** There is a unit circle that starts out painted white. Every second, you choose uniformly at random an arc of arclength 1 of the circle and paint it a new color. You use a new color each time, and new paint covers up old paint. Let c_n be the expected number of colors visible after n seconds. Compute $\lim_{n \to \infty} c_n$.