CMIMC 2020

1. In a game of ping-pong, the score is $4-10$. Six points later, the score is $10-10$. You remark that it was impressive that I won the previous 6 points in a row, but I remark back that you have won n points in a row. What the largest value of n such that this statement is true regardless of the order in which the points were distributed?

   Proposed by Cody Johnson

   Answer: 2

   Solution. We claim the answer is $n = \boxed{2}$. To prove this, first note that $n = 2$ is an upper bound, arising from the sequence of points

   AABAABAABAABAA.

   (Here A denotes a point from you while B denotes a point from me.) To see that $n = 2$ is sufficient, observe that my four wins divide your ten wins into five winning streaks; since the total number of wins is 10, some winning streak must have size at least two. This proves the claim.

2. Find all sets of five positive integers whose mode, mean, median, and range are all equal to 5.

   Proposed by David Altizio

   Answer: $2, 5, 5, 6, 7$; $3, 4, 5, 5, 8$

   Solution. When listing the five positive integers in order from least to greatest, the first, third, and fifth numbers are a, 5, and $a+5$, for some integer a between 1 and 5. Furthermore, since the mode of the data set is 5, either the second or fourth integer is 5, and in either case the remaining integer is some integer b. Since the mean of this data set is 5, we have

   $$a + 5 + 5 + b + (a + 5) = 25,$$

   or $2a + b = 10$. The only two solutions to this that satisfy $a \leq b \leq a + 5$ are $(2, 6)$ and $(3, 4)$; these correspond to the sets $\boxed{2, 5, 5, 6, 7 \text{ and } 3, 4, 5, 5, 8}$.

3. Let ABC be a triangle with centroid G and $BC = 3$. If ABC is similar to GAB, compute the area of ABC.

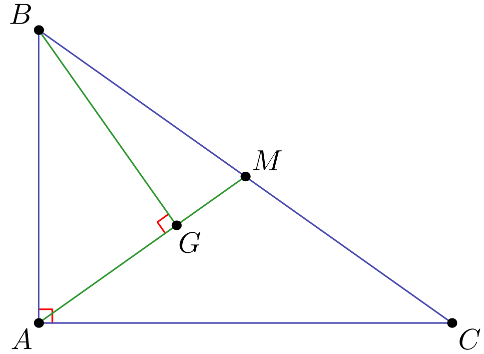
   Proposed by Howard Halim

   Answer: $\frac{3\sqrt{2}}{2}$

   Solution. Let M be the midpoint of $\overline{BC}$. The condition $\triangle ABC \sim \triangle GAB$ implies

   $$\angle MAB \equiv \angle GAB = \angle ABC,$$

   so $AM = MB = MC$. This implies $AB \perp AC$.



Now note that $GA = \frac{1}{3}BC = 1$, so upon letting $AB = x$ we obtain the equation $\frac{1}{x} = \frac{x}{3}$; this yields $x = \sqrt{3}$. Thus Pythagorean Theorem yields $AC = \sqrt{6}$, so the area of triangle ABC is $\frac{1}{2} \cdot \sqrt{3} \cdot \sqrt{6} = \boxed{\frac{3\sqrt{2}}{2}}$.

4. Given $n = 2020$, sort the 6 values

$$n^{n^2},\ 2^{2^{2^n}},\ n^{2^n},\ 2^{2^{n^2}},\ 2^{n^n},\ \text{and } 2^{n^{2^2}}$$

from **least** to **greatest**. Give your answer as a 6 digit permutation of the string "123456", where the number i corresponds to the i-th expression in the list, from left to right.

Proposed by Adam Bertelli

Answer: 163542

Solution. We can first simplify all of the expressions by taking the base 2 logarithm of each, giving us

$$n^2 \log n, 2^{2^n}, 2^n \log n, 2^{n^2}, n^n, n^4$$

Since logarithmic functions grow slower than polynomial functions, which in turn grow slower than exponentials, we immediately know that $1, 6$ are the two smallest values, in that order. For the remaining 4 values, we can take the base 2 logarithm again, giving:

$$2^n, n + \log\log n, n^2, n \log n$$

From here, we see that 3 is linear (plus a very slow-growing function), 5 is linear times a logarithm, 4 is quadratic, and 2 is exponential, giving us the remaining ordering of $3 < 5 < 4 < 2$, so our total order is 163542.

5. We say that a binary string s *contains* another binary string t if there exist indices $i_1, i_2, \ldots, i_{|t|}$ with $i_1 < i_2 < \ldots < i_{|t|}$ such that

$$s_{i_1} s_{i_2} \ldots s_{i_{|t|}} = t.$$

(Tn other words, t is found as a not necessarily contiguous substring of s.) For example, 110010 contains 111. What is the length of the shortest string s which contains the binary representations of all the positive integers less than or equal to 2048?

Proposed by Misha Ivkov

Answer: 22

Solution. The shortest such string is $(10)^{11}$ (repeat 10, eleven times). Notice that $2048 = 2^{11}$ so there must be a 1 followed by 11 zeroes as a non-contiguous substring. Furthermore, 2047 is just a sequence of 11 ones so we need at least 11 ones.

Now, every binary integer < 2047 has length at most 11 so we choose one of the two options from each of the 11 blocks of size 2 to represent it (adding zeroes on the front, optionally.

6. Misha is currently taking a Complexity Theory exam, but he seems to have forgotten a lot of the material! In the question, he is asked to fill in the following boxes with $\subseteq$ and $\subsetneq$ to identify the relationship between different complexity classes:

$$\mathsf{NL} \ \square\ \mathsf{P} \ \square\ \mathsf{NP} \ \square\ \mathsf{PH} \ \square\ \mathsf{PSPACE} \ \square\ \mathsf{EXP}$$

and

$$\mathsf{coNL} \ \square\ \mathsf{P} \ \square\ \mathsf{coNP} \ \square\ \mathsf{PH}$$

Luckily, he remembers that $\mathsf{P} \neq \mathsf{EXP}$, $\mathsf{NL} \neq \mathsf{PSPACE}$, $\mathsf{coNL} \neq \mathsf{PSPACE}$, and $\mathsf{NP} \neq \mathsf{coNP} \implies \mathsf{P} \neq \mathsf{NP} \wedge \mathsf{P} \neq \mathsf{coNP}$. How many ways are there for him to fill in the boxes so as not to contradict what he remembers?

Proposed by Misha Ivkov

Answer: 89

Solution. More is known: $\mathsf{NL} = \mathsf{coNL}$ (nondeterministic space classes closed under complement by Immerman–Szelepcsényi) but that would make the problem less fun.

Now, let's case on $\mathsf{NP} \neq \mathsf{coNP}$. If this is true, then we have pretty much everything else for free: the inclusions NL ? P, coNL ? P, NP ? PH, coNP ? PH, PH ? PSPACE, PSPACE ? EXP are all free to be anything. So, in this case there are $2^6 = 64$ possibilities.

Now suppose that $\mathsf{NP} = \mathsf{coNP}$. Let's case on PH ? PSPACE. If $\mathsf{PH} \neq \mathsf{PSPACE}$ then all other assertions are again free. These are NL ? P, coNL ? P, NP ? PH, PSPACE ? EXP so there are $2^4 = 16$ possibilities.

Finally, suppose that $\mathsf{PH} = \mathsf{PSPACE}$. Then the only constraints left are NL ? P, coNL ? P, P ? PH, PSPACE ? EXP. Notice that if $\mathsf{P} = \mathsf{PH}$ then all the others are fixed to be not equal, while if $\mathsf{P} \neq \mathsf{PH}$ then all the others are free. So, the total number of possibilities is $64 + 16 + 8 + 1 = 89$.

7. Points P and Q lie on a circle ω. The tangents to ω at P and Q intersect at point T, and point R is chosen on ω so that T and R lie on opposite sides of PQ and $\angle PQR = \angle PTQ$. Let RT meet ω for the second time at point S. Given that $PQ = 12$ and $TR = 28$, determine PS.

Proposed by David Altizio

Answer: $\frac{36}{7}$

Solution. First observe that angles TPQ and PRQ are congruent since they both subtend minor arc PQ; combining this with $\angle PQR = \angle PTQ$ means that $\triangle PTQ \sim \triangle PQR$, and in particular that $RQ = PQ = 12$. This means that $\frac{PT}{PQ} = \frac{PQ}{PR}$, so $PT \cdot PR = 144$. But now invoking the fact that triangles PTS and RTP are similar yields

$$PS = \frac{PT \cdot PR}{TR} = \frac{144}{28} = \frac{36}{7}.$$

8. Simplify

$$\binom{2020}{1010}\binom{1010}{1010} + \binom{2019}{1010}\binom{1011}{1010} + \cdots + \binom{1011}{1010}\binom{2019}{1010} + \binom{1010}{1010}\binom{2020}{1010}.$$

Proposed by Adam Bertelli

Answer: $\binom{3031}{1010}$ **OR** $\binom{3031}{2021}$

Solution. We present two solutions to this problem.

First Solution, by Double Counting We claim, in general, that

$$\sum_{k \geq 0} \binom{k}{n}\binom{m-k}{n} = \binom{m+1}{2n+1}.$$

Indeed, both sides count the number of $2n+1$-element subsets of $m+1$; the right hand side does it directly, while the left hand side cases on the location of the median element in the set. This means the sum simplifies to $\boxed{\binom{3031}{1010}}$.

Second solution, by Generating Functions Recall the power series expansion

$$\frac{1}{(1-x)^k} = \sum_{n \geq 0} \binom{n}{k} x^k$$

for every positive integer k. Thus, we may use the formula for convolution to write

$$\sum_{m+n=3030}\binom{m}{1010}\binom{n}{1010}=[x^{3030}]\left(\sum_{k=0}^{\infty}\binom{k}{1010}x^k\right)^2=[x^{1010}]\left(\sum_{k=0}^{\infty}\binom{k+1010}{1010}x^k\right)^2$$
$$=[x^{1010}]\left(\frac{1}{(1-x)^{1011}}\right)^2=[x^{1010}]\frac{1}{(1-x)^{2022}}=\boxed{\binom{3031}{2021}=\binom{3031}{1010}}.$$

9. Over all natural numbers n with 16 (not necessarily distinct) prime divisors, one of them maximizes the value of $\frac{s(n)}{n}$, where $s(n)$ denotes the sum of the divisors of n. What is the value of $d(d(n))$, where $d(n)$ is the the number of divisors of n?

   Proposed by Adam Bertelli

   Answer: 54

   Solution. Recall that we can quickly compute the sum of divisors of a number $n=\prod p_i^{k_i}$ as $\prod(1+p_i+\cdots+p_i^{k_i})$, as, when expanded, this product covers all possible choices for how many factors of p_i a given divisor can have. Thus, when we add a new factor of p_i, $s(n)$ is multiplied by $\frac{1+\cdots+p_i^{k_i+1}}{1+\cdots+p_i^{k_i}}$, hence $\frac{s(n)}{n}$ is multiplied by a factor of $\frac{1+\cdots+p_i^{k_i+1}}{p_i(1+\cdots+p_i^{k_i})}=1+\frac{1}{p_i+\cdots+p_i^{k_i+1}}$. Since we are trying to increase the value of $\frac{s(n)}{n}$ as quickly as possible through adding 16 prime factors, it suffices to list out small values of $p_i+\cdots+p_i^{k_i+1}$, and pick the smallest one each time (as this will correspond to the largest possible $1+\frac{1}{p_i+\cdots+p_i^{k_i+1}}$ at each step). We can do this by hand quite easily:

2 :	$\boxed{2}$	$\boxed{6}$	$\boxed{14}$	$\boxed{30}$	62	$\cdots$
3 :	$\boxed{3}$	$\boxed{12}$	39	$\cdots$		
5 :	$\boxed{5}$	$\boxed{30}$	$\cdots$			
7 :	$\boxed{7}$	56				
11 :	$\boxed{11}$	$\cdots$				
13 :	$\boxed{13}$	$\cdots$				
17 :	$\boxed{17}$	$\cdots$				
19 :	$\boxed{19}$	$\cdots$				
23 :	$\boxed{23}$	$\cdots$				
29 :	$\boxed{29}$	$\cdots$				
31 :	$\boxed{31}$	$\cdots$				
37 :	37	$\cdots$				
$\cdots$						

   This tells us our optimal choice of n has factors of $2^4, 3^2, 5^2$, and 8 other primes chosen exactly once, hence $d(n)=(4+1)(2+1)^2(1+1)^8=2^83^25$, so $d(d(n))=(8+1)(2+1)(1+1)=54$.

10. Let ABC be a triangle. The incircle ω of $\triangle ABC$, which has radius 3, is tangent to $\overline{BC}$ at D. Suppose the length of the altitude from A to $\overline{BC}$ is 15 and $BD^2+CD^2=33$. What is BC?

    Proposed by Daniel Li

    Answer: $3\sqrt{7}$

    Solution. We'll solve the problem for general values of h (the height from A), r (the inradius of $\triangle ABC$), and d (the value of BD^2+CD^2).

    First let $\alpha := h/r$; via an area argument or the angle bisector theorem, compute $b+c=(\alpha-1)a$. This means $s=\frac{\alpha}{2}a$, so

$$\frac{r}{r_a}=\frac{s-a}{s}=\frac{\alpha-2}{\alpha},$$

whence $r_a = \frac{r\alpha}{\alpha-2} = \frac{rh}{h-2r}$. (For geometry experts, this also follows from the fact that $(A, X; I, I_A) = -1$, where X is the foot of the A-angle bisector.)

We now claim a lemma of independent interest.

Lemma 1. *In any triangle ABC, $rr_a = BD \cdot CD$.*

Proof. Let E be the tangency point of the A-circle with BC. Then triangles BID and I_AEB are similar, which yields the desired equality. □

Therefore
$$a = \sqrt{(BD+CD)^2} = \sqrt{d + 2rr_a} = \sqrt{d + \frac{2r^2h}{h-2r}}.$$

Plugging in the specific numbers yields an answer of $\boxed{3\sqrt{7}}$.

11. Find the number of ordered triples of integers (a, b, c), each between 1 and 64, such that
$$a^2 + b^2 \equiv c^2 \pmod{64}.$$

Proposed by Misha Ivkov

Solution. Let f be the map $x \mapsto x^2$ on $\mathbb{Z}/64$.

The set of residues which are $1 \mod 2$ map surjectively under f onto the set Y_1 of residues which are 1 mod 8; this map is 4-to-1. Given this, the set of residues which are $2 \mod 4$ map surjectively to the set Y_2 of residues $4 \mod 32$, this map is 8-to-1. The residues which are $4 \mod 8$ all map to 16, and the residues which are $0 \mod 8$ all map to 0; these are both 8-to-1.

We case on the possibilities for (a^2, b^2, c^2). It is easy to see that the possible residue types are $(X, 0, X)$ (where X is any square mod 64) and $(Y_1, 16, Y_1)$, where we have undercounted by a factor of 2 except when $X = 0$. We deal with the $(X, 0, X)$ case first.

- **Case 1:** $X = 0$. There are then 8 choices for each of a, b, c, all of which work, so there are 8^3 choices here.
- **Case 2:** $X \in Y_1$. There are 32 possible choices for a here. There are 8 choices for b. With a fixed, we must consider all c which square to a^2; since the squaring map is 4-to-1 on Y_1, there are 4 choices here. The total count is $32 \cdot 8 \cdot 4 \cdot 2$, where we multiply by 2 to account for switching the first two coordinates.
- **Case 3:** $X \in Y_2$. With the same logic as in Case 2, there are 16 choices for a here, 8 choices for b, and 8 choices for c since f is 8-to-1 from $2 \mod 4 \to 4 \mod 32$. This gives $16 \cdot 8 \cdot 8 \cdot 2$.
- **Case 4:** $X = 16$. The same argument gives $8 \cdot 8 \cdot 8 \cdot 2$.

Lastly, we need to count the solutions of the form $(Y_1, 16, Y_1)$; this is parallel to Case 2 and contributes $32 \cdot 8 \cdot 4 \cdot 2$ solutions. So the answer is
$$8^3 + 32 \cdot 8 \cdot 4 \cdot 2 + 16 \cdot 8 \cdot 8 \cdot 2 + 8 \cdot 8 \cdot 8 \cdot 2 + 32 \cdot 8 \cdot 4 \cdot 2,$$
or
$$2^9 + 2^{11} + 2^{11} + 2^{10} + 2^{11} = \boxed{7680}.$$

12. Determine the maximum possible value of
$$\sqrt{x}(2\sqrt{x} + \sqrt{1-x})(3\sqrt{x} + 4\sqrt{1-x})$$
over all $x \in [0, 1]$.

Answer: $4\sqrt{5}$

Solution. Notice that the points $(\sqrt{x}, \sqrt{1-x})$ across $x \in [0,1]$ are just the points (a,b) in the first quadrant lying on the unit circle. Then we are trying to maximize

$$a(2a+b)(3a+4b)$$

subject to $a^2+b^2=1$ and $a,b \geq 0$. Observe that $5a+(3a+4b)=4(2a+b)$. Thus we have by AM-GM:

$$5a(3a+4b) \leq 4(2a+b)^2$$

and so if P is our desired product, then

$$5P \leq 4(2a+b)^3$$

with equality when $5a = 3a+4b$, or $a = 2b$. But we observe that $2a+b$ is also maximized subject to $a^2+b^2=1$ when $a = 2b$ (consider that $2a+b$ is the dot product $(a,b)\cdot(2,1)$), so we can assume that $a=2b$ and so $2a+b=\sqrt{2^2+1^2}=\sqrt{5}$. Thus we get $P \leq \frac{4}{5}\cdot(\sqrt{5})^3 = 4\sqrt{5}$ with equality when $a=2b$, which occurs when $x=\frac{4}{5}$.

13. Given 10 points arranged in a equilateral triangular grid of side length 4, how many ways are there to choose two distinct line segments, with endpoints on the grid, that intersect in exactly one point (not necessarily on the grid)?

Proposed by Adam Bertelli

Answer: 519

Solution. We consider two separate cases: when the endpoints of the two segments consist of 3 points in total, or 4 points in total.

In the first case, for any selection of 3 points we make, there are 3 ways to choose two segments that will satisfy this property, unless the 3 points are collinear, in which case there is only one way, by choosing each half of the line as its own segment. The number of ways to have 3 points collinear is $3\cdot\binom{4}{3}+3\cdot\binom{3}{3}=15$, and the number of total ways to choose 3 points is $\binom{10}{3}=120$, so the total in this case is $3(120-15)+15=330$.

In the second case, if we choose 4 points that form a non-concave boundary with positive area, then there is exactly one way to have the segments intersect, by choosing the two diagonals of this shape (the other two choices will give opposite sides). Thus we only have to subtract off the choices where all 4 points are collinear (degenerate), which there are 3 of, or when a point is strictly contained in a triangle formed by the other 3. Clearly the only point that could satisfy this property is the center point, and we can count 5 possible shapes of triangles, pictured below, with $1,2,3,6,6$ ways to rotate/reflect them, respectively, giving us a total of

$$\binom{10}{4}-3-(1+2+3+6+6)=210-21=\boxed{189}.$$

14. Let $a_0=1$ and for all $n\geq 1$ let a_n be the smaller root of the equation

$$4^{-n}x^2-x+a_{n-1}=0.$$

Given that a_n approaches a value L as n goes to infinity, what is the value of L?

Answer: $\frac{\pi^2}{4}$

Solution. We see that

$$a_{n-1}=a_n(1-4^{-n}a_n)$$

If we let $a_n=4^n\sin^2\theta_n$ for $\theta_n\in[0,\pi/2]$ then we get

$$\sin^2\theta_{n-1}=4\sin^2\theta_n\cos^2\theta_n=\sin^2 2\theta_n$$

Now because a_n is chosen to be the smaller root we are forced to have $\theta_n=\theta_{n-1}/2$. Since $a_0=1$, we have $\theta_0=\pi/2$. It follows that

$$a_n=4^n\sin^2\frac{\pi}{2^{n+1}}=\left(2^n\sin\frac{\pi}{2\cdot 2^n}\right)^2$$

Now it is well-known that

$$\lim_{x\to 0}\frac{\sin(ax)}{x}=a$$

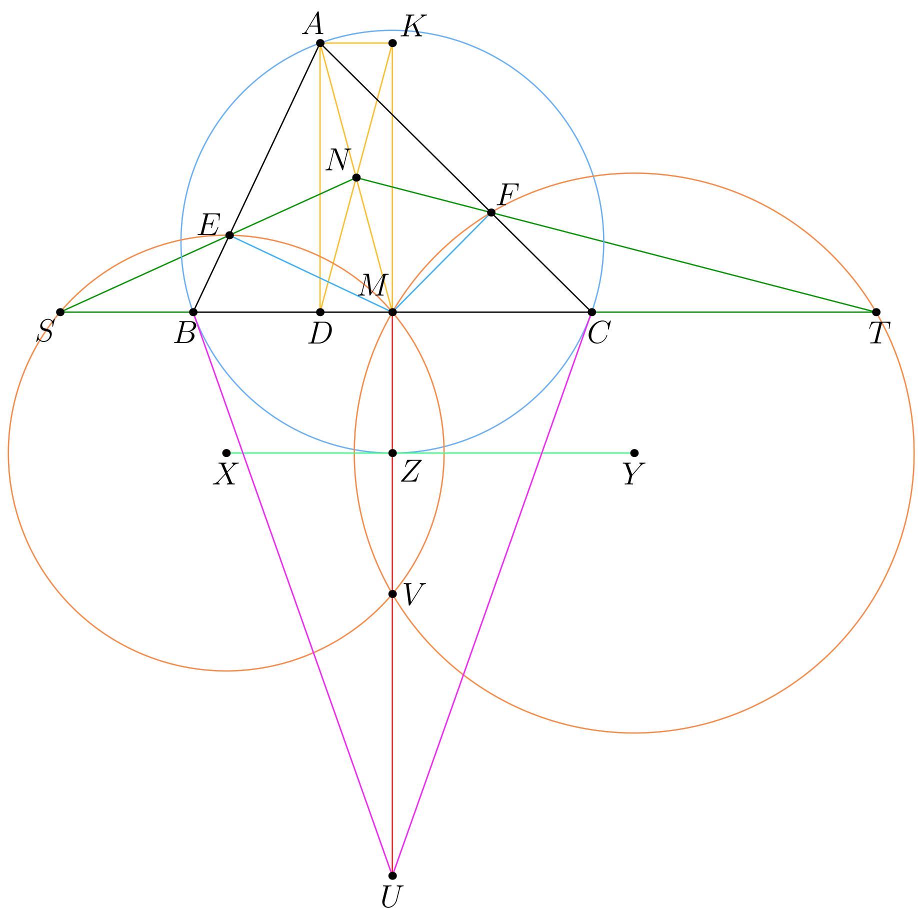
for any a. Applying this to $a=\pi/2$ along the sequence $x=2^{-n}$ shows that $a_n\to\pi^2/4$.

15. Let ABC be an acute triangle with $AB=3$ and $AC=4$. Suppose M is the midpoint of segment $\overline{BC}$, N is the midpoint of $\overline{AM}$, and E and F are the feet of the altitudes of M onto $\overline{AB}$ and $\overline{AC}$, respectively. Further suppose BC intersects NE at S and NF at T, and let X and Y be the circumcenters of $\triangle MES$ and $\triangle MFT$, respectively. If XY is tangent to the circumcircle of $\triangle ABC$, what is the area of $\triangle ABC$?

Proposed by Gunmay Handa

Answer: $4\sqrt{2}$

Solution. Let circles $\odot(MES)$ and $\odot(MFT)$ meet again at V, and Z be the intersection of MV and XY (which is also the midpoint of $\overline{MV}$). Let D be the foot of the altitude from A to BC, and define K be such that $ADMK$ is a rectangle.



Claim 1. *The lines MV and BC are perpendicular.*

Proof. First remark that

$$-1=(B,C;M,\infty)\stackrel{A}{=}(E,F;M,K).$$

Let the tangents to $(AEMF)$ at E and F meet at V'. Note that $\angle SEV'=\angle TFV'=90^\circ$ because N is the center of $\odot(AEMF)$. We also have V',M,K collinear, so $V'M\perp BC$. Therefore V' is on $\odot(MES)$ and $\odot(MFT)$ by right angles so $V'=V$. $\square$

Let the tangents to $\odot(ABC)$ at B and C meet at U.

Claim 2. *Point V is the midpoint of $\overline{MU}$.*

Proof. Let H be the orthocenter of ABC, A' be the antipode of A on $\odot(ABC)$, and F_1, E_1 be the feet of the altitudes from B and C respectively. Note that $\odot(AH)$ (the circle with diameter $\overline{AH}$), $(AEMF)$, and (ABC) concur at some point P because H, M, A' are collinear and we have similar figures

$$PE_1F_1HM \sim PEFMV \sim PBCA'U.$$

Since M is the midpoint of $\overline{HA'}$, V is the midpoint of $\overline{MU}$. □

Now let R be the circumradius and $x = \frac{OM}{R}$. Then $\frac{OU}{R} = \frac{1}{x}$ (by inversion). Since V is the midpoint of $\overline{MU}$ and Z is the midpoint of $\overline{MV}$, we have $\frac{OZ}{R} = \frac{3}{4}x + \frac{1}{4}\frac{1}{x}$. By the first claim, XY must be tangent to $\odot(ABC)$ at Z, so $OZ = R$. Therefore $x = \frac{1}{3}$. From here we easily get $\sin A = \frac{\sqrt{8}}{3}$, and so

$$[ABC] = \frac{1}{2}AB \cdot AC \sin A = \frac{1}{2} \cdot 3 \cdot 4 \cdot \frac{\sqrt{8}}{3} = \boxed{4\sqrt{2}}.$$