CMIMC 2020

Team Round

INSTRUCTIONS

1. Do not look at the test before the proctor starts the round.
2. This test consists of 15 short-answer problems to be solved in 60 minutes.
3. Write your team name and team ID on your answer sheet.
4. Write your answers in the corresponding boxes on the answer sheets.
5. No computational aids other than pencil/pen are permitted.
6. If you believe that the test contains an error, submit your protest in writing to Doherty Hall 2302.

Team

1. In a game of ping-pong, the score is $4-10$. Six points later, the score is $10-10$. You remark that it was impressive that I won the previous 6 points in a row, but I remark back that you have won n points in a row. What the largest value of n such that this statement is true regardless of the order in which the points were distributed?

2. Find all sets of five positive integers whose mode, mean, median, and range are all equal to 5.

3. Let ABC be a triangle with centroid G and $BC = 3$. If ABC is similar to GAB, compute the area of ABC.

4. Given $n = 2020$, sort the 6 values

$$n^{n^2},\ 2^{2^{2^n}},\ n^{2^n},\ 2^{2^{n^2}},\ 2^{n^n},\ \text{and } 2^{n^{2^2}}$$

from **least** to **greatest**. Give your answer as a 6 digit permutation of the string "123456", where the number i corresponds to the i-th expression in the list, from left to right.

5. We say that a binary string s *contains* another binary string t if there exist indices $i_1, i_2, \ldots, i_{|t|}$ with $i_1 < i_2 < \ldots < i_{|t|}$ such that

$$s_{i_1} s_{i_2} \ldots s_{i_{|t|}} = t.$$

(Tn other words, t is found as a not necessarily contiguous substring of s.) For example, 110010 contains 111. What is the length of the shortest string s which contains the binary representations of all the positive integers less than or equal to 2048?

6. Misha is currently taking a Complexity Theory exam, but he seems to have forgotten a lot of the material! In the question, he is asked to fill in the following boxes with $\subseteq$ and $\subsetneq$ to identify the relationship between different complexity classes:

$$\mathsf{NL} \ \Box\ \mathsf{P} \ \Box\ \mathsf{NP} \ \Box\ \mathsf{PH} \ \Box\ \mathsf{PSPACE} \ \Box\ \mathsf{EXP}$$

and

$$\mathsf{coNL} \ \Box\ \mathsf{P} \ \Box\ \mathsf{coNP} \ \Box\ \mathsf{PH}$$

Luckily, he remembers that $\mathsf{P} \neq \mathsf{EXP}$, $\mathsf{NL} \neq \mathsf{PSPACE}$, $\mathsf{coNL} \neq \mathsf{PSPACE}$, and $\mathsf{NP} \neq \mathsf{coNP} \implies \mathsf{P} \neq \mathsf{NP} \wedge \mathsf{P} \neq \mathsf{coNP}$. How many ways are there for him to fill in the boxes so as not to contradict what he remembers?

7. Points P and Q lie on a circle ω. The tangents to ω at P and Q intersect at point T, and point R is chosen on ω so that T and R lie on opposite sides of PQ and $\angle PQR = \angle PTQ$. Let RT meet ω for the second time at point S. Given that $PQ = 12$ and $TR = 28$, determine PS.

8. Simplify

$$\binom{2020}{1010}\binom{1010}{1010} + \binom{2019}{1010}\binom{1011}{1010} + \cdots + \binom{1011}{1010}\binom{2019}{1010} + \binom{1010}{1010}\binom{2020}{1010}.$$

9. Over all natural numbers n with 16 (not necessarily distinct) prime divisors, one of them maximizes the value of $\frac{s(n)}{n}$, where $s(n)$ denotes the sum of the divisors of n. What is the value of $d(d(n))$, where $d(n)$ is the the number of divisors of n?

10. Let ABC be a triangle. The incircle ω of $\triangle ABC$, which has radius 3, is tangent to $\overline{BC}$ at D. Suppose the length of the altitude from A to $\overline{BC}$ is 15 and $BD^2 + CD^2 = 33$. What is BC?

11. Find the number of ordered triples of integers (a, b, c), each between 1 and 64, such that

$$a^2 + b^2 \equiv c^2 \pmod{64}.$$

12. Determine the maximum possible value of

$$\sqrt{x}(2\sqrt{x}+\sqrt{1-x})(3\sqrt{x}+4\sqrt{1-x})$$

over all $x \in [0,1]$.

13. Given 10 points arranged in a equilateral triangular grid of side length 4, how many ways are there to choose two distinct line segments, with endpoints on the grid, that intersect in exactly one point (not necessarily on the grid)?

14. Let $a_0 = 1$ and for all $n \geq 1$ let a_n be the smaller root of the equation

$$4^{-n}x^2 - x + a_{n-1} = 0.$$

Given that a_n approaches a value L as n goes to infinity, what is the value of L?

15. Let ABC be an acute triangle with $AB = 3$ and $AC = 4$. Suppose M is the midpoint of segment $\overline{BC}$, N is the midpoint of $\overline{AM}$, and E and F are the feet of the altitudes of M onto $\overline{AB}$ and $\overline{AC}$, respectively. Further suppose BC intersects NE at S and NF at T, and let X and Y be the circumcenters of $\triangle MES$ and $\triangle MFT$, respectively. If XY is tangent to the circumcircle of $\triangle ABC$, what is the area of $\triangle ABC$?

16. (Estimation) Choose a point (x, y) in the square bounded by $(0,0),(0,1),(1,0)$ and $(1,1)$. Your score is the minimal distance from your point to any other team's submitted point. Your answer must be in the form $(0.abcd, 0.efgh)$ where a, b, c, d, e, f, g, h are decimal digits.