NYCMT 2025-2026 Homework #3

NYCMT

September 26 - October 24, 2025

These problems are due October 24th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve.

Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on October 24th, you can scan your solutions and email them to jamespapaelias@gmail.com, kylewu32@gmail.com, and sjschool26@gmail.com. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or use one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

Problem 1. We define $\varphi(n)$ to be the number of positive integers $k \leq n$ such that $\gcd(k, n) = 1$.

(a) Note that 1 and 2 are positive integers n such that for any positive integer k with $k \leq n$ and $\varphi(k) \mid \varphi(n)$, it follows that $k \mid n$. Do there exist any other such n?

(b) Note that $1, 2$, and 6 are positive integers n such that for any positive integer k with $k \leq n$ and $\varphi(k) \mid \varphi(n)$, it follows that $k \mid n^2$. Do there exist any other such n?

Problem 2. Let $\triangle ABC$ be a triangle, and let P, Q, and R be points on BC, CA, and AB, respectively. Prove that the triangle formed by the centers of the circumcircles of $\triangle AQR$, $\triangle BRP$, and $\triangle CPQ$ is similar to ABC.

Problem 3. The *Fibonacci sequence* $F_1, F_2, \ldots$ satisfies $F_1 = 1$, $F_2 = 1$, and $F_{n+1} = F_n + F_{n-1}$. Given that the Fibonacci number F_m is a prime for some positive integer m, find in terms of m the smallest positive integer k such that $F_m \mid F_{n+k} - F_n$ for all positive integers n.

Problem 4. Suppose $f(x)$ is a monic quadratic polynomial such that there exists an increasing arithmetic sequence $x_1 < x_2 < x_3 < x_4$ where $|f(x_1)| = |f(x_2)| = |f(x_3)| = |f(x_4)| = 2020$. Compute the absolute difference of the two roots of $f(x)$.

Problem 5. Every second while on a grid, Frankie Focus can move forward one square, turn 90° in any direction, or wait. However, Frankie Focus is very alert and chooses not to go in the same direction twice in a row (where turns and waits are ignored). Find the minimum number of copies of Frankie Focus that must be placed on a 2025×2025 grid so that after a finite amount of time every square has been occupied by exactly one copy of Frankie Focus exactly once.

Alternate Formulation: Frankie Focus can move along the orthogonal directions in a square grid. However, he is very alert and chooses not to move in the same direction twice. Define a *focused path* to be a set of squares that Frankie may traverse in a finite number of moves without going on the same square twice. What is the minimum number of disjoint focused paths needed to exactly cover a 2025×2025 grid? (Here, by "exactly cover" we mean no focused path leaves the grid.)