NYCMT 2025-2026 Homework #2

NYCMT

September 19 - September 26, 2025

These problems are due September 26th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve.

Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on September 26th, you can scan your solutions and email them to jamespapaelias@gmail.com, kylewu32@gmail.com, and sjschool26@gmail.com. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or use one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

Problem 1. Prove that every positive integer can be expressed uniquely as the sum of one or more Fibonacci numbers, no two of which have consecutive indices. (For example, 67 has the unique representation $67 = 55 + 8 + 3 + 1$.)

Problem 2. Compute

$$\tan\left(\frac{\pi}{36}\right)\tan\left(\frac{\pi}{72}\right)+\tan\left(\frac{3\pi}{36}\right)\tan\left(\frac{3\pi}{72}\right)+\cdots+\tan\left(\frac{71\pi}{36}\right)\tan\left(\frac{71\pi}{72}\right),$$

where the sum is taken over numerators $k\pi$ for odd k between 1 and 71.

Problem 3. Let $F(n)$ denote the smallest positive integer greater than n whose sum of digits is equal to to the sum of digits of n. For example, $F(2025) = 2034$. Compute $F(1) + F(2) + F(3) + \cdots + F(1000)$.

Problem 4. Let H be the orthocenter of acute triangle $\triangle ABC$. Show that $BH = 2R\cos(\angle ABC)$, where R is the circumradius of $\triangle ABC$.

Problem 5. Suis draws a convex n-gon and n circles in the plane, one with a diameter on each side of the n-gon. She then fills in the circles (including boundary) to create n possibly overlapping disks in the same positions, and inquires if she has covered every point in the n-gon's interior with the disks.

(a) Prove that if $n = 3$, Suis always covers the entire interior.

(b) Prove that if $n = 4$, Suis always covers the entire interior.

(c) Prove that if $n = 5$, Suis covers the entire interior only if the pentagon formed by connecting the centers of the circles has some angle which measures at most $90°$.