NYCMT 2025-2026 Homework #1 Solutions

NYCMT

September 2025

Problem 1. Let $a_1, a_2, a_3, \ldots$ be an infinite sequence of positive integers with the property that the product of any set of $k \geq 1$ consecutive terms is not a 2025th power. Prove that there exist infinitely many primes which divide some element of the sequence $\{a_i\}$.

Solution. We say that for an integer n and a prime p, we have that $\nu_p(n) = k$ if k is the largest integer such that $p^k \mid n$.

Suppose for the sake of contradiction that for some natural number n, only n primes $p_1, p_2, \ldots, p_n$ divided an element of the sequence $\{a_i\}$. Let P_k denote the product of the first k terms of the sequence $\{a_i\}$. It follows by definition that $P_a \mid P_b$ for all $b \geq a$. Consider taking the sequence $\{V_i\}$ of ordered n-tuples given by

$$V_i = (\nu_{p_1}(P_i), \nu_{p_2}(P_i), \ldots, \nu_{p_n}(P_i)) \pmod{2025}$$

where the reduction mod 2025 is taken component-wise. Since there are only 2025^n possibilities for V_i, we get by the Pigeonhole Principle that there exist positive integers a and b such that $b \geq a$ and $V_a = V_b$. It follows that $\frac{P_b}{P_a}$ is a perfect 2025th power, which can also be written as product of consecutive terms of $\{a_i\}$:

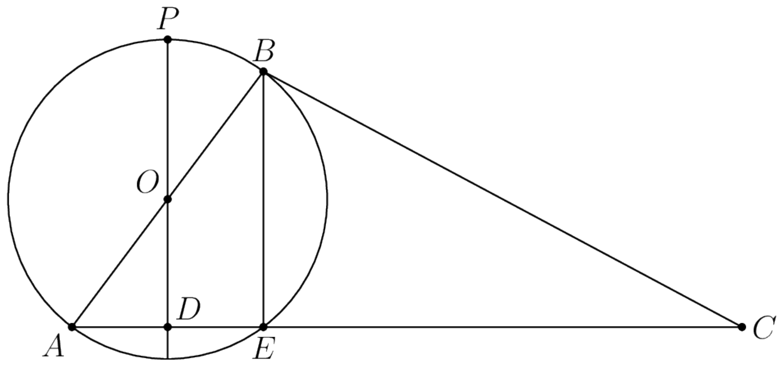
$$\frac{P_b}{P_a} = \prod_{i=a+1}^{b} a_i.$$

Thus, we have obtained a contradiction and it follows that there exist infinitely many primes which divide some element of the sequence $\{a_i\}$, as desired. □

Problem 2. Triangle ABC has $AB = 10$, $BC = 17$, and $CA = 21$. Point P lies on the circle with diameter AB. What is the greatest possible area of APC?

Answer. $\boxed{\frac{189}{2}}$

Solution. Let O be the midpoint of segment AB. Maximizing the area of $\triangle APC$ is equivalent to maximizing the distance from P to AC. Thus, we choose P to be the furthest point on the circle to AC, which is the intersection of circle O and the perpendicular to AC through O that is on the same side of AC as B. Let D be the intersection of line OP and line AC. We wish to find the length of DP, the height of $\triangle APC$.



Let E be the foot of the altitude from B onto AC. Since OD and BE are both perpendicular to AC, they are parallel to each other and thus $\angle ABE = \angle AOD$. Since $\angle OAD = \angle BAE$, we find that by AA, $\triangle OAD \sim \triangle BAE$. Since

$$\frac{OD}{BE} = \frac{AO}{AB} = \frac{1}{2},$$

we find that $OD = BE/2$, so it is sufficient to find the length of BE. By Heron's Formula,

$$[ABC] = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{24 \cdot 14 \cdot 7 \cdot 3} = 84.$$

Since we also know that $[ABC] = \frac{1}{2} \cdot AC \cdot BE$, we obtain the equation $84 = \frac{1}{2} \cdot 21 \cdot BE$, so we can solve for $BE = 8$, so $OD = 4$. Thus,

$$DP = DO + OP = 4 + \frac{1}{2}AB = 4 + 5 = 9,$$

so the greatest possible area of APC is $\frac{1}{2} \cdot 21 \cdot 9 = \boxed{\frac{189}{2}}$.

□

Problem 3. A positive integer n is called *internet-enabled* if the binary representation of n^2 contains exactly two 1's. Find the sum of the first 5 *internet-enabled* numbers.

Answer. $\boxed{93}$

Solution. We consider squares of the form $2^a + 2^b$ for nonnegative integers $a > b$. If b is odd then note that $\nu_2(2^a + 2^b)$ is odd and thus it cannot be a square. If b is even, consider factoring out 2^b so that the number becomes $2^b(2^{a-b} + 1)$. Then $2^{a-b} + 1 = k^2$ for some integer k implying

$$2^{a-b} = (k + 1)(k - 1).$$

Thus, $k + 1$ and $k - 1$ are powers of 2 which differ by two, which only has the solution $k = 3$. Thus, $2^{a-b} + 1 = 9$ and $a - b = 3$. This means the five smallest solutions are $n^2 = 2^{b+3} + 2^b$ for each $b = 0, 1, \ldots, 5$, giving $n = 3, 6, 12, 24, 48$ with a sum of $\boxed{93}$. $\square$

Problem 4. Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$ and $BC = AD = 10$ and $CD = 20$. Let Ω be the circumcircle of $ABCD$, and let M be the midpoint of AB. Line CM intersects Ω at $X \neq C$ and DX intersects AB at Y. Given that $AY = 3$, compute the length of side AB.

Answer. $\boxed{15}$

Solution 1. Note that since $\triangle XMY$ maps to $\triangle XCD$ under a homothety centered at X, the circumcircle ω_1 of $\triangle XMY$ is internally tangent to the circumcircle Ω if $\triangle XCD$. Also, note that if DM intersects Ω at $X' \neq X$, then $X'X \parallel AB$ by symmetry so $\triangle DYM$ maps to $\triangle DXX'$ under a homothety centered at X. Thus, the circumcircle ω_2 of DYM is internally tangent to the circumcircle Ω of DXX'. Now, from Radical Axis on the circles Ω, ω_1, ω_2, we find that the tangents to Ω at D and X meet on line AB at a point Z. By the parallel condition, $\angle ZAD = \angle ADC$. By the angle condition for tangency, $\angle ZDA = \angle ACD$. Thus we get $\triangle ZAD \sim \triangle ADC$, giving $ZA = 5$ by similarity. Let x denote half of the desired length AB. Using power of a point from Z to both Ω and ω_1, we get:

$$XZ^2 = 5(5 + 2x) = 8(x + 5)$$

which we solve to get $2x = \boxed{15}$, our desired length. $\square$

Solution 2. Alternatively, we can get the central concurrence as follows. Let P_∞ denote the point at infinity with respect to parallel lines AB and CD. It follows that $(A, B; M, P_\infty) = -1$. Projecting onto the circle Ω from C, it follows that $AXBD$ is a harmonic quadrilateral, and in particular, that the tangents to Ω at D and X concur at a point Z on AB. From here, the solution proceeds as above, or alternatively by noting that $(Z, Y; A, B) = -1$ and computing the cross ratio expression in terms of YB. $\square$

Problem 5. Let $n \geq 2$ be a positive integer and let $z_1, \ldots, z_n$ be nonzero complex numbers satisfying $\overline{z_k} + \frac{1}{z_k} = 2z_{k+1}$ for each $1 \leq k \leq n$, where indices are taken cyclically. Find, in terms of n, all possible values of $(z_1, z_2, \ldots, z_n)$.

Answer. $(1, 1, \ldots, 1)$ or $(-1, -1, \ldots, -1)$ when n is odd, and $(z, z^{-1}, z, \ldots, z^{-1})$ for $|z| = 1$ when n is even.

Solution. Taking magnitudes of each equation, we find that $|z_k| + \frac{1}{|z_k|} = 2|z_{k+1}|$. Thus, if we let $f(x) = \frac{x+1/x}{2}$, then $f(|z_k|) = |z_{k+1}|$, so $f^n(|z_1|) = |z_1|$, where $f^n(x)$ denotes n applications of $f(x)$.
Now, note that if $x > 1$ then $1 < f(x)$ by AM-GM and $f(x) < x$ since $\frac{1}{x} < x \implies \frac{x+1/x}{2} \leq x$. Thus, if we suppose that $|z_1| > 1$, then we have

$$|z_1| > f(|z_1|) > f^2(|z_1|) > \cdots > f^n(|z_1|) > 1,$$

contradicting the fact that $f^n(|z_1|) = |z_1|$. Therefore $|z_1| \leq 1$, and by similar reasoning $|z_2| \leq 1$. Also, if we suppose that $0 < |z_1| < 1$, then

$$f(|z_1|) = \frac{|z_1| + 1/|z_1|}{2} = f(1/|z_1|) > 1$$

so $|z_2| = f(|z_1|) > 1$, a contradiction.
Thus $|z_1| = 1$, and $\overline{z_1} = \frac{1}{z_1}$ so we get that $\overline{z_1} + \frac{1}{z_1} = \frac{2}{z_1} = 2z_2$ so $z_2 = \frac{1}{z_1}$. By analogous reasoning, $z_{k+1} = \frac{1}{z_k}$ for each $1 \leq k \leq n$.
If n is odd, we find that this implies $z_1 = \frac{1}{z_1}$ so that $z_1 = \pm 1$ so $(z_1, z_2, \ldots, z_n) = (1, 1, \ldots, 1)$ or $(-1, -1, \ldots, -1)$. If n is even, then any z_1 with $|z_1| = 1$ works, and we find that the answer is $(z_1, z_2, \ldots, z_n) = (z, z^{-1}, z, z^{-1}, \ldots, z^{-1})$. Thus, the answer is

$$\boxed{\begin{cases} (1, 1, \ldots, 1) \text{ or } (-1, -1, \ldots, -1) \text{ when } n \text{ is odd,} \\ (z, z^{-1}, z, \ldots, z^{-1}) \text{ for } |z| = 1 \text{ when } n \text{ is even.} \end{cases}}$$

□