1. [**5**] Two line segments $\overline{BR}$ and $\overline{UM}$ intersect at a point O such that $BO \times RO = UO \times MO$. Find $\angle BUR + \angle RMB$ in degrees.

   Proposed by: Christina Stepin

   Answer: $\boxed{180^\circ}$

   Solution: The intersection ratio makes $BRUM$ a cyclic quadrilateral, so by definition, the sum of these angles must be 180°.

2. [**5**] A 5-digit number contains no repeating digits. If the product of its digits is 5670, find the sum of its digits.

   Proposed by: Amber Bajaj

   Answer: $\boxed{30}$

   Solution: The prime factorization of 5670 is $5670 = 2 \times 3^4 \times 5 \times 7$. 5 and 7 must be digits in the original number, which leaves 3 additional digits. To make use of all the 3s, and produce a unique set of digits, one 3 must be kept as a 3, two 3s will form a pair $3 \times 3 = 9$ and one 3 must combine with another factor (only 2 is a possibility). Thus, the digits of the number are: 5, 7, 3, 6, 9. The sum of these digits is 30.

3. [**5**] A scientist drops a ball from a height of 243 centimeters. Each time it bounces back up, it reaches a maximum height $\frac{2}{3}$ of what it was before. Eventually, the ball's peak is below 40 centimeters, and the now bored scientist walks away to make a sandwich. Right as it reaches that last observed peak (which is below 40 centimeters), how much distance has the ball traveled?

   Proposed by: Jaeho Lee

   Answer: $\boxed{1055}$

   Solution: The heights of the peaks are $243, 162, 108, 72, 48$, and 32. Except for the first and last peak, the ball travels from the ground up to the peak and back down. The ball starts at the first peak and ends at the last peak, so these distances are only counted once. The answer is $243 + 2 \cdot (162 + 108 + 72 + 48) + 32 = 1055$.

4. [**6**] Find the sum of the coefficients of $(xy + 2x + 3y)^4$.

   Proposed by: Joshua Kou

   Answer: $\boxed{1296}$

   Solution: Setting $x = y = 1$ yields the sum of the values of the coefficients. Thus the answer is $(1 + 2 + 3)^4 = 1296$.

5. [**6**] Bruno the bear sits in a circle with his friends Alice, Bob, Charlie, David, and Eve. If Bruno refuses to sit next to Bob, how many ways are there to seat the group of 6 friends? (Rotations of a seating arrangement are considered to be the same.)

   Proposed by: Amy Qiao

   Answer: $\boxed{72}$

   Solution: There are $5! = 120$ ways to seat the 6 friends in a circle. We count the number of ways for Bob and Bruno to sit next to each other. Fix Bruno's seat. Then Bob must sit in one of the two seats next to Bruno, and there are $4! = 24$ ways to seat the other four friends. Therefore, there are $2 \cdot 24 = 48$ ways for Bob and Bruno to sit next to each other, so there are $120 - 48 = 72$

ways to seat the friends such that Bob and Bruno are not adjacent.

Alternative solution: Seat the 6 friends randomly. The probability that Bob is not in one of the two positions adjacent to Bruno is $\frac{3}{5}$, so there are $120 \cdot \frac{3}{5} = 72$ ways for them to sit.

6. [**6**] A point on the xy-plane is called "nice" if both of its coordinates are rational numbers. Given a convex 2025-gon P inscribed in a circle O whose center is NOT "nice", what is the maximum number of "nice" vertices of P?

   Proposed by: Byron Zou

   Answer: $\boxed{2}$

   Solution: It's easy to construct examples of 2025-gons with two "nice" vertices. For example, one can consider a regular 2025-gon whose first two vertices are $(0, 0)$ and $(1, 0)$ and inscribed in a circle with an irrational radius.

   To show that we cannot have more than two "nice" vertices, notice that if all three vertices of a triangle are all "nice", then the circumcenter of the triangle must be "nice". This is because given a triangle ABC where A, B, C are all "nice". Denote D to be the midpoint of AB and E to be the midpoint of BC. Then, D and E are both "nice". Let l be the perpendicular bisector of AB and m be the perpendicular bisector of BC. Then, l and m can be expressed with linear functions in x, y whose coefficients are all rational. Therefore, their intersection must be "nice," but the intersection is precisely the circumcenter of triangle ABC, which contradicts the circumcenter is NOT "nice."

7. [**7**] Let $h(n)$ be defined as the greatest n-digit number, all of whose digits are prime, that is the product of n distinct prime numbers. Compute $h(4)$.

   Proposed by: Adeethyia Shankar

   Answer: $\boxed{7755}$

   Solution: We check the largest possible 4-digit numbers with 4 prime digits. 7777 is not the product of 4 distinct prime numbers as its prime factorization is $7777 = 7 \cdot 1111 = 7 \cdot 11 \cdot 101$. 7775 also does not work as it is divisible by $25 = 5^2$, which is two of the same prime. We can check that 7757 does not work because it is not divisible by any prime less than 10 and $10^4 > 7757$. However, $7755 = 11 \cdot 705 = 11 \cdot 3 \cdot 5 \cdot 47$, so $h(4) = 7755$.

8. [**7**] a is chosen randomly on the interval $[0, 4]$, and b is chosen randomly on the interval $[0, 2]$. What is the probability that $1 \leq |a - b| \leq 2$?

   Proposed by: Jake Rosenberg

   Answer: $\boxed{\frac{5}{16}}$

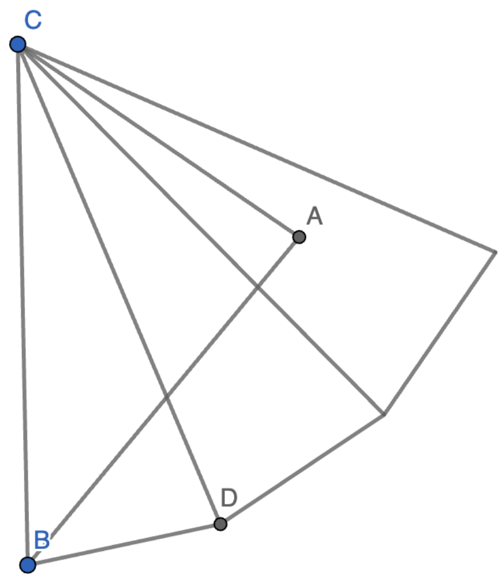
   Solution: We can plot the point (a, b) on a graph. Clearly (a, b) must lie inside the rectangle bounded the lines $x = 0$, $x = 4$, $y = 0$, and $y = 2$. If $1 \leq |a - b| \leq 2$, then (a, b) either lies in between the lines $y = x + 1$ and $y = x + 2$ or lies in between the lines $y = x - 1$ and $y = x - 2$. The area of the region bounded by the rectangle,
   $y = x + 1$, and $y = x + 2$ is $\frac{1}{2}$. The area of the region bounded by the rectangle,
   $y = x - 1$, and $y = x - 2$ is 2. Therefore the probability that $1 \leq |a - b| \leq 2$ is $\frac{1/2\,+\,2}{8} = \frac{5}{16}$.

9. [**7**] Consider a room shaped like a hexagonal pyramid with base side length 2 and height $2\sqrt{6}$. A beetle is currently at the centroid of one of the walls. If it may only crawl among the walls and floor of the room, how long is its shortest path to one of the base vertices of the wall opposite to the one it's on right now?

Proposed by: Jaeho Lee

Answer: $\boxed{\dfrac{2\sqrt{229}}{7}}$

Solution: *Note: this question was voided due to an error in the answer key. Originally, the answer was listed as* $2\sqrt{7}$*, which is incorrect.*



Let A be the centroid of one of the faces, B be the base vertex on the opposite wall that is closer when measured clockwise, and C be the apex of the pyramid. After dropping an altitude from C to the base of the pyramid, we can use the Pythagorean theorem to compute the slant height to be $3\sqrt{3}$. It follows that the distance from C to the centroid A is $\frac{2}{3} \cdot 3\sqrt{3} = 2\sqrt{3}$.

There are two possible paths the beetle can take: crawling along the walls or crawling along the floor. We first compute the shortest path if the beetle crawls along the walls.

If the beetle crawls along the walls, then the shortest path is the length of AB in the diagram above. Using the Pythagorean theorem, we can compute

$$BC = \sqrt{\left(\frac{1}{2} \cdot \text{base}\right)^2 + \text{height}^2} = \sqrt{1^2 + (3\sqrt{3})^2} = 2\sqrt{7},$$

so we just need $\cos\angle ACB$. Let D be the vertex adjacent to B that is closer to A and define $\alpha = \angle BCD$. We can compute $\cos\left(\frac{\alpha}{2}\right) = \frac{3\sqrt{3}}{2\sqrt{7}}$. Using the double angle formula, we have

$$\cos\alpha = 2\cos^2\left(\frac{\alpha}{2}\right) - 1 = 2 \cdot \frac{27}{28} - 1 = \frac{13}{14}.$$

Using the double angle formula again, we have

$$\cos(2\alpha) = 2\cos^2\alpha - 1 = 2 \cdot \frac{169}{196} - 1 = \frac{71}{98}.$$

Finally, we use the cosine angle addition formula to get

$$\cos\left(2\alpha + \frac{\alpha}{2}\right) = \frac{3\sqrt{3}}{2\sqrt{7}} \cdot \frac{71}{98} - \sqrt{1 - \left(\frac{3\sqrt{3}}{2\sqrt{7}}\right)^2} \cdot \sqrt{1 - \left(\frac{71}{98}\right)^2}$$

$$= \frac{213\sqrt{3} - \sqrt{(2\sqrt{7})^2 - (3\sqrt{3})^2} \cdot \sqrt{98^2 - 71^2}}{2\sqrt{7} \cdot 98} = \frac{213\sqrt{3} - 39\sqrt{3}}{2\sqrt{7} \cdot 98} = \frac{174\sqrt{3}}{2\sqrt{7} \cdot 98} = \frac{87\sqrt{21}}{686}.$$

We compute

$$AB^2 = AC^2 + BC^2 - 2 \cdot AC \cdot BC \cos\left(2\alpha + \frac{\alpha}{2}\right) = 12 + 28 - 2 \cdot \sqrt{12} \cdot \sqrt{28} \cdot \frac{87\sqrt{21}}{686}$$

$$= 40 - 8\sqrt{21} \cdot \frac{87\sqrt{21}}{686} = \frac{916}{49}.$$

This gives us $AB = \frac{2\sqrt{229}}{7}$.

Now, we consider the case where the beetle crawls along the floor. Note that the centroid A is $\sqrt{3}$ units above the base of the triangle. If we flatten the wall so that it is on the same plane as the floor, we see that the centroid is $2\sqrt{3} + \sqrt{3}$ units away in the y direction and 1 unit away in the x direction, so the shortest path in this case has length $2\sqrt{7}$.

Since $\frac{2\sqrt{229}}{7} < \frac{2 \cdot 16}{7} < 2 \cdot 2.5 < 2\sqrt{6.25} < 2\sqrt{7}$, we can conclude that the shortest path has length $\frac{2\sqrt{229}}{7}$.

10. [**8**] How many ways can you write $19 = a + b + c + d + e$, where a, b, c, d, and e are distinct positive integers?

    Proposed by: Ethan Bove

    Answer: $\boxed{600}$

    Solution: Assume to start that $a < b < c < d < e$. Since we require the integers to be nonnegative, we have that $a \geq 1$, $b \geq 2$, $c \geq 3$, $d \geq 4$ and $e \geq 5$. We can therefore normalize and write $4 = a' + b' + c' + d' + e'$, where $a' = a - 1$, $b' = b - 2$ and so on. Solutions to our original equation with $a < b < c < d < e$ will correspond exactly to solutions for this equation with $a' \leq b' \leq c' \leq d' \leq e'$. One can count that the only such solutions are:

    $$4 = 0 + 4 = 0 + 1 + 3 = 0 + 2 + 2 = 0 + 1 + 1 + 2 = 1 + 1 + 1 + 1.$$

    Thus, there are five choices for a', b', c', d' and e'. Since we assumed they were increasing to begin with, to get the total count, we multiply this by $5!$, yielding $5 \cdot 5! = 600$.

11. [**8**] Let $a_1, a_2, a_3, \ldots$ be a geometric sequence of real numbers with $a_n \neq 0$ for all positive integers n. Given that $a_3 = 1$, find the minimum possible value of $a_1 + 2a_2 + 2a_4 + a_5$.

    Proposed by: Leo Zhang

Answer: $\boxed{-2}$

Solution: Let r be the common ratio such that $a_{n+1} = a_n r$ for $n \geq 1$. Since $a_3 = 1$, we find that $a_1 = \frac{1}{r^2}, a_2 = \frac{1}{r}, a_4 = r$, and $a_5 = r^2$. Then

$$a_1 + 2a_2 + 2a_4 + a_5 = \frac{1}{r^2} + \frac{2}{r} + 2r + r^2.$$

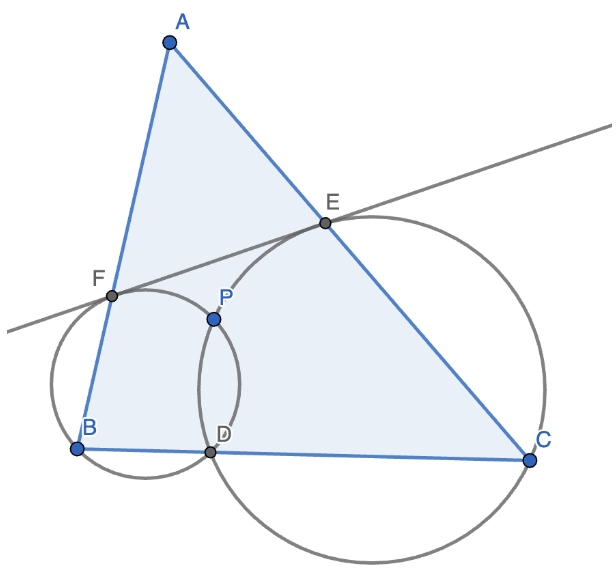
Let $z = r + \frac{1}{r}$. Then this expression is equal to $z^2 + 2z - 2 = (z+1)^2 - 3$. This is minimized when $z + 1$ is as close to 0 as possible. Note that $|z| \geq 2$ by the AM-GM inequality, so this achieves a minimum value of -2 when $z = -2$, which corresponds to $r = -1$.

12. [**8**] Let P be a point in the interior of $\triangle ABC$. Let D be a point on BC such that the circumcircles of $\triangle CDP$ and $\triangle BDP$ intersect AC and AB at E and F, respectively. Suppose that EF is tangent to both circumcircles. Given that $\angle A = 55°$, compute $\angle BPC$ in degrees.

Proposed by: Max Liu

Answer: $\boxed{110°}$

Solution:



Let $\beta = \angle PBC$ and $\gamma = \angle PCB$. Then $\angle B - \beta = \angle PBF = \angle PFE$ and $\angle C - \gamma = \angle PCE = \angle PEF$. Since

$$\begin{aligned}\angle EPF &= 360° - \angle DPF - \angle DPE \\ &= 360° - (180° - \angle B) - (180° - \angle C) \\ &= \angle B + \angle C \\ &= 180° - \angle A,\end{aligned}$$

$AFPE$ is cyclic, so $\angle PFE + \angle PEF = 180 - \angle EPF = \angle A$. Therefore, $\angle B - \beta + \angle C - \gamma = \angle A$. Adding $\angle A$ to both sides, we get $\angle A + \angle B + \angle C - \beta - \gamma = 2\angle A$. The LHS is just $180° - \beta - \gamma = \angle BPC$, so the answer is $2 \cdot \angle A = 110°$.

13. [**9**] Find the product of all real numbers $x > 0$ for which

$$\log_2\left(\frac{1}{x^2}\right) + \log_x(8) = \left(\frac{5}{\log_x(4) + 3\log_2(x)}\right).$$

Proposed by: Amber Bajaj

Answer: $\boxed{1}$

Solution: Let $a = \log_2(x)$. Then, $\frac{1}{a} = \log_x(2)$. Plugging these in, we have:

$$-2a + \frac{3}{a} = \frac{5}{\frac{2}{a} + 3a}$$

$$-6a^2 + 5 + \frac{6}{a^2} = 5$$

Multiplying both sides by a^2:

$$-6a^4 + 5a^2 + 6 = 5a^2$$

$$-6a^4 + 6 = 0$$

$$-6(a^4 - 1) = 0$$

$$-6(a^2 + 1)(a + 1)(a - 1) = 0$$

The first root is imaginary, thus $a = \pm 1$.

To solve for x, we have $\pm 1 = \log_2(x)$, which yields $x = 2^1 = 2$ and $x = 2^{-1} = \frac{1}{2}$. The product of these two solutions is $\frac{1}{2} \times 2 = 1$.

14. [**9**] Define a function $A(n)$ below for all integers n

$$A(n) = \begin{cases} A(A(n+5)) & \text{if } n \le 1000 \\ n - 2 & \text{if } n > 1000 \end{cases}$$

What is the value of $A(5)$?

Proposed by: Alice Wang

Answer: $\boxed{999}$

Solution: By repeating the first definition of the function, we get

$$A(5) = A^{201}(1005)$$

We can easily notice that for any $n \in \mathbb{Z}$, $n > 4$, we have

$$A^n(1005) = A^{n-3}(999) = A^{n-2}(1004) = A^{n-4}(1000) = A^{n-3}(1005)$$

and thus

$$A^{201}(1005) = A^3(1005) = 999$$

15. [**9**] We fill each cell of an 8×8 table with a number from $\{1, 2, 3, 4, 5, 6\}$. How many ways are there to fill the table such that the sum of numbers in each row is divisible by 2 and the sum of numbers in each column is divisible by 3?

Proposed by: An Cao

Answer: $\boxed{6^{56}}$

Solution: First, we can fill the first 7 rows and 7 columns with any number. This gives us 6^{49} possibilities. For the first 7 cells of the last column, fill in each of these with a number that makes the sum of numbers in the corresponding rows divisible by 2. Each of these cells has 3 options, so there are 3^7 possibilities.

For the last row, fill in each of the first 7 cells with a number that makes the sum of numbers in the corresponding columns divisible by 3. There are 2 options for each cell, so there are 2^7 possibilities.

For the one remaining cell, we have to make the last row divisible by 2 and last column divisible by 3, and there is only 1 option. In total, there are $6^{49} \cdot 3^7 \cdot 2^7 = 6^{56}$ ways to fill in the table.

16. [**10**] Let the arithmetic and harmonic means of the areas of three distinct similar triangles be 42 and $\frac{96}{7}$, respectively. Given that the second largest triangle has area and perimeter 24, find the arithmetic mean of the perimeters of the three triangles.

Proposed by: Jerry Sun

Answer: $\boxed{28}$

Solution: Let the areas be a_1, b_1, and c_1, where $a_1 \leq b_1 \leq c_1$. Similarly, let the perimeters be a_2, b_2, and c_2, where $a_2 \leq b_2 \leq c_2$. We are given that $a_1 + b_1 + c_1 = 126$ and $\frac{3}{\frac{1}{a_1}+\frac{1}{b_1}+\frac{1}{c_1}} = \frac{96}{7}$. The second equation can be rewritten as $\frac{1}{a_1} + \frac{1}{b_1} + \frac{1}{c_1} = \frac{21}{96}$. Plugging in $b_1 = b_2 = 24$, this yields $a_1 + c_1 = 102$ and $\frac{1}{a_1} + \frac{1}{c_1} = \frac{21}{96} - \frac{1}{24} = \frac{17}{96}$. We rewrite the last equation as $\frac{a_1+c_1}{a_1c_1} = \frac{17}{96}$, so $a_1c_1 = \frac{96}{17} \cdot 102 = 96 \cdot 6$. We can solve the quadratic or notice that $a_1 = 6, c_1 = 96$ is a solution.

Now, we compute a_2 and c_2. Since the area is proportional to the squared perimeter, let $\text{Area} = k \cdot \text{Perimeter}^2$ for this set of similar triangles. Then $k = \frac{1}{24}$, so $a_2 = \sqrt{24 \cdot 6} = 12$ and $c_2 = \sqrt{24 \cdot 96} = 48$. The arithmetic mean of the perimeters is $\frac{12+24+48}{3} = 28$.

17. [**10**] Let $ABCD$ be an isosceles trapezoid with $AB = 12, CD = 28$, and $AD = BC = 17$. Let Ω be the circle with diameter BC, and let $\overline{AC}$ intersect Ω at $K \neq C$. Compute AK.

Proposed by: Max Liu

Answer: $\boxed{\frac{48}{5}}$

Solution: Let $\overline{AB}$ intersect Ω again at E. Since BC is a diameter of Ω, we know that $\angle BEC = 90°$. Therefore, CE is the height of the trapezoid. By symmetry, $BE = \frac{CD-AB}{2} = 8$, so $CE = \sqrt{17^2 - 8^2} = 15$. We can then compute $AC = \sqrt{(12+8)^2 + 15^2} = 25$. By power of a point, $AB \cdot AE = AK \cdot AC \implies AK = \frac{AB \cdot AE}{AC} = \frac{12 \cdot 20}{25} = \frac{48}{5}$.

18. [**10**] A box has 11 slips of paper in it each labeled with a different integer between 1 and 11, inclusive. Carla picks four random slips of paper from the box, without replacement, and then

arranges them in ascending order. What is the probability that, once she has finished arranging the slips of paper, two consecutive slips have a difference of exactly 5?

Proposed by: Sebastian Weinberger

Answer: $\boxed{\frac{2}{11}}$

Solution: The gap is either $1-6, 2-7, 3-8, 4-9, 5-10$, or $6-11$. Since it is impossible to have multiple such gaps, we need only to sum the total number of ways to get each possible gap (unordered) and divide by the total number of unordered possible combinations of papers. The number of ways to obtain the gap $1-6$ is the number of ways to pull $1, 6$, and two other slips of paper between 7 and 11 inclusive. There are a total of $\binom{5}{2}$ ways to do the latter and only one to do the former. This same logic works for each case, so we get a total of $6 \times \binom{5}{2}$ "good" combinations out of a total of $\binom{11}{4}$. After canceling, this is just $\frac{2}{11}$.

19. [**11**] Suppose $x^3 - 5x + 2 = 0$ has 3 distinct complex roots a, b, c. Compute $a^4 + b^4 + c^4$.

Proposed by: An Cao

Answer: $\boxed{50}$

Solution: Since a is a root of $x^3 - 5x + 2 = 0$, so $a^3 = 5a - 2$ and thus $a^4 = a(5a-2) = 5a^2 - 2a$. Similarly for b^4, c^4, we have

$$\begin{aligned} a^4 + b^4 + c^4 &= 5(a^2 + b^2 + c^2) - 2(a + b + c) \\ &= 5(a + b + c)^2 - 10(ab + bc + ca) - 2(a + b + c) \end{aligned} \tag{1}$$

By Vieta's formulas, we have

$$a + b + c = 0$$

$$ab + bc + ca = -5$$

Thus $a^4 + b^4 + c^4 = -10 \cdot -5 = 50$.

20. [**11**] Define $p(x) = (1 + x + x^2 + \cdots + x^{203})^3$. Expand $p(x)$ into the form $a_0 + a_1 x + a_2 x^2 + \cdots + a_{609} x^{609}$. Find a_{300}.

Proposed by: Leo Zhang

Answer: $\boxed{\binom{302}{2} - 3\binom{98}{2} \text{ or } 31192}$

Solution: The problem is equivalent to: find the number of ordered pairs (x, y, z) such that $x + y + z = 300$ and $0 \leq x, y, z \leq 203$.

Using stars and bars with 300 stars and 2 bars, we find that there are $\binom{302}{2}$ pairs (x, y, z) that satisfy $x + y + z = 300$ and $x, y, z \geq 0$. Consider the triples that do not satisfy $x, y, z \leq 203$. Clearly, there is only one number that is greater than 203, so without loss of generality assume that $z \geq 204$. Let $w = z - 204$. Then, $w \geq 0$ and $x + y + w = 96$. There are $\binom{98}{2}$ pairs of nonnegative solutions (x, y, w). By symmetry, there are $3\binom{98}{2}$ pairs of nonnegative solutions that does not satisfy $x, y, z \leq 203$, so the total number of ordered pairs (x, y, z) is $\binom{302}{2} - 3\binom{98}{2} = 31192$. Therefore, we must have $a_{300} = 31192$.

21. [**11**] Choose six distinct vertices of a regular nonagon. What is the expected number of distinct equilateral triangles that can be drawn using only the chosen vertices?

Proposed by: Sebastian Weinberger

Answer: $\boxed{\dfrac{5}{7}}$

Solution: Label the vertices 1 through 9. There are no equilateral triangles if all three unchosen vertices are different (mod 3), one if exactly two of the three are the same (mod 3), and two if all three unchosen vertices are the same (mod 3). We can count $3 \cdot 3 \cdot 3 = 27$ ways to choose the three unchosen vertices such that they are all different modulo see, since this is the number of ways to choose a vertex from $\{1, 4, 7\}$, one from $\{2, 5, 8\}$, and one from $\{3, 6, 9\}$. It is also not hard to see that there are only three cases where they are all the same. The total number of possible choices is $\binom{9}{3} = 84$, so by complementary counting there are $84 - 3 - 27 = 54$ choices where two are the same and one is different. This gives us an expected value of $\frac{0 \cdot 27 + 1 \cdot 54 + 2 \cdot 3}{84} = \frac{5}{7}$.

22. [**12**] There is a classroom with chairs arranged in 4 rows of 6 chairs facing the front of the classroom. Bruno wants to make a seating chart for his 23 students. However, he is worried that students might be blocked by taller students sitting in chairs in front of the student, in the same column of chairs.

If Bruno selects a seating arrangement uniformly at random, what is the expected number of students who won't be blocked by a taller one? Here, assume that each student has a unique height.

Proposed by: Luke Choi

Answer: $\boxed{\dfrac{49}{4}}$

Solution: The key observation is that there will be 5 columns of 4 students with distinct height, and there will be one column of chairs with an empty seat and 3 students of distinct height. By linearity of expectation, we can compute the expected number of unblocked students in each column and sum over them.

A student is unblocked if and only if they are taller than all the students sitting in front of them. Since we choose arrangements uniformly at random, the probability that a student with i students sitting ahead is unblocked is $1/(i + 1)$. Therefore, in a full column of 4 students, the expected number of unblocked students is

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12}$$

Similarly, in a column of 3 students, this is

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

Here, we can just ignore the empty seat since it can block no students. Therefore, our answer is

$$5 \cdot \frac{25}{12} + \frac{11}{6} = \boxed{\frac{49}{4}}$$

23. [**12**] Given that a and b are non-negative numbers and $3a+b=10$, find the minimum value of $\sqrt{9a^2+16}+\sqrt{b^2+9}$.

Proposed by: Haoyang Xu

Answer: $\boxed{\sqrt{149}}$

Solution: Let $x=3a$, then $b=10-x$. We want to find the minimum value of

$$y=\sqrt{x^2+16}+\sqrt{(10-x)^2+9}$$

for $0\le x\le 10$. The geometric meaning of $\sqrt{x^2+16}$ is the distance between $(x,0)$ and $(0,4)$, and the geometric meaning of $\sqrt{(10-x)^2+9}$ is the distance between $(x,0)$ and $(10,-3)$. By the triangle inequality, for any three points A,B,C, we have $|AB|+|BC|\ge|AC|$. Let $A=(0,4),B=(x,0),C=(10,-3)$. The distance between A and C is

$$\sqrt{(10-0)^2+(-3-4)^2}=\sqrt{149}$$

So the minimum value of y is $\sqrt{149}$, and equality holds when the point $(3a,0)$ lies on the straight line segment joining $(0,4)$ and $(10,-3)$.

24. [**12**] Let ABC be a triangle with $AB=1$. Let l be the line passing through A parallel to $\overline{BC}$. Suppose that there is a circle passing through C tangent to $\overline{BC}$, $\overline{AB}$, and l. Compute the maximum possible area of $\triangle ABC$.

Proposed by: Max Liu

Answer: $\boxed{\frac{3\sqrt{3}}{16}}$

Solution: Let O be the center of the circle and r be its radius. Let $\theta=\angle OBC=\angle OBA$. Then $\tan\theta=\frac{r}{BC}$. Since the circle is tangent to l and $\overline{BC}$, we know that the height of the altitude from A to $\overline{BC}$ is $2r$, so $\sin 2\theta=\sin\angle ABC=\frac{2r}{AB}$. Dividing this by the first equation, we have

$$\frac{\sin 2\theta}{\tan\theta}=\frac{\frac{2r}{AB}}{\frac{r}{BC}}=\frac{2BC}{AB}.$$

Using the double angle formula twice, the left-hand side becomes $\frac{2\sin\theta\cos\theta}{\frac{\sin\theta}{\cos\theta}}=2\cos^2\theta=\cos 2\theta+1$. Now, we can express $\sin B=\sin 2\theta$ in terms of AB and BC. We know that

$$\cos 2\theta=\frac{\sin 2\theta}{\tan\theta}-1=\frac{2BC}{AB}-1,$$

so

$$\sin 2\theta=\sqrt{1-\cos^2 2\theta}=\sqrt{1-\frac{4BC^2}{AB^2}+\frac{4BC}{AB}-1}=2\sqrt{\frac{BC}{AB}-\frac{BC^2}{AB^2}}.$$

The area of $\triangle ABC$ is

$$\frac{1}{2}AB\cdot BC\sin B=\sqrt{AB\cdot BC^3-BC^4}.$$

We are given that $AB=1$, so we just want to maximize $\sqrt{BC^3-BC^4}$. Since $f(x)=\sqrt{x}$ is increasing, we just need to find BC that maximizes $BC^3-BC^4=BC^3(1-BC)$. We write this as $27\cdot\left[\frac{BC}{3}\cdot\frac{BC}{3}\cdot\frac{BC}{3}\cdot(1-BC)\right]$. Note that the sum of the four terms being multiplied is constant, so by the AM-GM inequality, this is maximized when $\frac{BC}{3}=(1-BC)\implies BC=\frac{3}{4}$. This yields a maximum area of $\sqrt{\frac{27}{64}-\frac{81}{256}}=\sqrt{\frac{27}{256}}=\frac{3\sqrt{3}}{16}$.

25. [**13**] Let a, b, and c be distinct real numbers such that

$$a = \frac{bc+2}{1-b-c}$$
$$b = \frac{2ac+1}{1-2a-2c}$$
$$c = \frac{3ab+4}{1-3a-3b}.$$

Compute $a+b+c$.

Proposed by: Max Liu

Answer: $\boxed{-\frac{7}{11}}$

Solution: We multiply out the denominator in each equation to get

$$a - ab - ac = bc + 2$$
$$b - 2ab - 2bc = 2ac + 1$$
$$c - 3ac - 3bc = 3ab + 4.$$

We can rearrange the equations so that the term $ab + bc + ca$ appears in all of them. We get

$$a = (ab + bc + ca) + 2$$
$$b = 2(ab + bc + ca) + 1$$
$$c = 3(ab + bc + ca) + 4.$$

Let $x = ab + bc + ca$. Then $a = x + 2, b = 2x + 1$, and $c = 3x + 4$. Plugging these back into the expression for x, we get

$$x = (x + 2)(2x + 1) + (2x + 1)(3x + 4) + (3x + 4)(x + 2) = 11x^2 + 26x + 14.$$

Therefore, $11x^2 + 25x + 14 = 0$. This factors as $(x + 1)(11x + 14) = 0$, so we have $x = -1$ or $x = -\frac{14}{11}$. If $x = -1$, then $a = 1, b = -1$, and $c = 1$, but these are not distinct. Plugging in $x = -\frac{14}{11}$ yields $a = \frac{8}{11}, b = -\frac{17}{11}$, and $c = \frac{2}{11}$, so $a + b + c = -\frac{7}{11}$.

26. [**13**] Nijika has a fair n sided die with $n \geq 4$, with each side containing one of the distinct integers from 1 to n inclusive. The average number of times she needs to roll for her to get m consecutive rolls of the same integer, with $m \geq 1$, is $97 - 3n$. What is the sum of all possible values of n?

Proposed by: Mason Yu

Answer: $\boxed{68}$

Solution: We show that the average number of times Nijika needs to roll is $1+n+n^2+\cdots+n^{m-1}$.

Proof: Let $E(i)$ be the average number of times Nijika needs to roll to get m consecutive rolls of the same integer, given she already has i consecutive rolls. We wish to find a closed form for $E(0)$. We have:

$$E(i) = 1 + \frac{1}{n}E(i+1) + \frac{n-1}{n}E(1) \text{ for } 1 \leq i \leq m-1$$

$$E(m) = 0$$

Rearranging, we get

$$E(i) - E(1) = 1 + \frac{1}{n}(E(i+1) - E(1)).$$

Let $F(i) = E(i) - E(1)$. Then $F(i) = 1 + \frac{1}{n}F(i+1)$, so $F(i+1) = nF(i) - n$.

Note that $F(1) = 0$ and $F(m) = -E(1)$. Therefore,

$$\begin{aligned} F(2) &= -n \\ F(3) &= -n^2 - n \\ &\cdots \\ F(i) &= -n^{i-1} - n^{i-2} - \cdots - n \end{aligned}$$

It follows that

$$F(m) = -n^{m-1} - n^{m-2} - \cdots - n = -E(1),$$

so

$$E(0) = 1 + E(1) = 1 + n + n^2 + \cdots + n^{m-1}.$$

We want to solve $1 + n + n^2 + \cdots + n^{m-1} = 97 - 3n$ for all integers m and n. By the geometric series formula, this simplifies to $\frac{n^m - 1}{n-1} = 97 - 3n$, or $n^m + 3n^2 - 100n + 96 = 0$.

We test small values of m and find the corresponding solutions to n.

$m = 1 : 3n^2 - 99n + 96 = 0$. This factors to $3(n - 32)(n - 1) = 0$, so $n = 32, m = 1$ works.

$m = 2 : 4n^2 - 100n + 96 = 0$. This factors to $4(n - 24)(n - 1) = 0$, so $n = 24, m = 2$ works.

$m = 3 : n^3 + 3n^2 - 100n + 96 = (n-1)(n^2 + 4n - 96) = (n-1)(n+12)(n-8) = 0$, so $n = 8, m = 3$ works.

$m = 4 : n^4 + 3n^2 - 100n + 96 = (n - 1)(n - 4)(n^2 + 5n + 24) = 0$, so $n = 4, m = 4$ works.

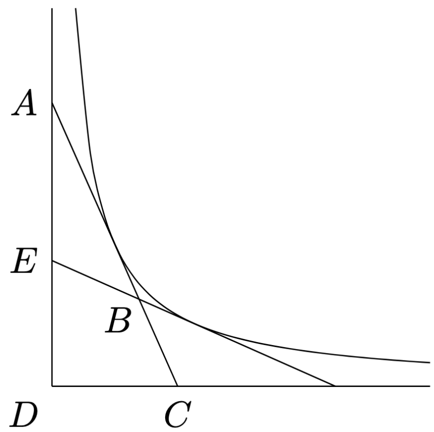
Note that n must decrease as m increases, because the number of consecutive rolls needed to get m identical rolls in a row increases as m increases. Therefore, since n is already minimized at $n = 4$, all possibilities of n are $32, 24, 8, 4$. Their sum is 68.

27. [**13**] Let ℓ be a line with slope less than -1 and positive y-intercept which is tangent to the hyperbola $y = 1/x$. Let Δ denote the triangle bounded by ℓ, the x axis, and the y axis. Let ℓ' denote the line obtained by reflecting ℓ across the line $y = x$. If ℓ' splits Δ into two regions of equal area, what is the x-intercept of ℓ?

Proposed by: Aiden Hennessey

Answer: $\boxed{\frac{2}{3}\sqrt{3}}$

Solution: Below is an illustration of the setup:



We wish to calculate the length DC given that ABE and $BCDE$ have the same area. Constructing segment DB, one sees $EBD \cong CBD$. Thus, we need ABE to have twice the area of EBD. Since these triangles share an altitude, it follows that the base of ABE must be twice as long, i.e. $AE = 2ED$. Since $ED = DC$, we conclude $AD = 3DC$.

The transformation $(x, y) \mapsto (\lambda x, y/\lambda)$ is area-preserving for all $\lambda > 0$. All triangles inscribed by the hyperbola can be brought to one another by such a transformation, so all triangles inscribed by the hyperbola have the same area. The triangle which is tangent to the hyperbola at $(1, 1)$ has area 2. Thus, $\frac{1}{2}(AD)(DC) = 2$. Substituting $AD = 3DC$ and solving gives $DC = \frac{2}{3}\sqrt{3}$.

28. [**14**] Let $\tau(n)$ be the number of divisors of n. Call an integer m a *tower* if $a^{\tau(a)} \not\equiv 1 \pmod{m}$ holds for all $2 \leq a \leq m - 1$. Given that there are five towers between 3 and 100 (inclusive), compute the sum of the five towers.

    Proposed by: Max Liu

    Answer: $\boxed{164}$

    Solution: Our general strategy is to choose specific values of a that eliminate many possible values of m.
    Suppose m is a tower. We pick $a = m - 1$. Then we must have $(m-1)^{\tau(m-1)} \not\equiv 1 \pmod{m}$. This implies that $\tau(m-1)$ is odd, so $m - 1$ must be a perfect square. Let $m - 1 = n^2$, so $m = n^2 + 1$. The possible values for m are $2^2 + 1, 3^2 + 1, \ldots, 9^2 + 1$.

    Since we are given that there are only five towers, we must be able to eliminate three of these values. For each of the possible towers $m = n^2 + 1$, we plug in $a = n$ and $a = n^2 + 1 - n$. This yields

$$n^{\tau(n)} \not\equiv 1 \pmod{n^2+1} \text{ and } (n^2+1-n)^{\tau(n^2+1-n)} \equiv (-n)^{\tau(n^2+1-n)} \not\equiv 1 \pmod{n^2+1}.$$

    Since $n^4 \equiv (-n)^4 \equiv 1 \pmod{n^2 + 1}$, we know that 4 cannot divide the exponents $\tau(n)$ and $\tau(n^2 + 1 - n)$. This eliminates $n = 5, 6,$ and 8 since $\tau(26 - 5) = \tau(21) = 4, \tau(6) = 4$, and $\tau(8) = 4$. Therefore, the five towers are $2^2 + 1, 3^2 + 1, 4^2 + 1, 7^2 + 1$, and $9^2 + 1$, so our answer is $5 + 10 + 17 + 50 + 82 = 164$.

29. [**14**] Let z be a complex number. Find the minimum possible value of $|z^2 - 4z + 9| + |z^2 + 4z + 9|$.

    Proposed by: Leo Zhang

Answer: $\boxed{8\sqrt{5}}$

Solution: Let point P represent z on the complex plane. Consider the points $A = (2, \sqrt{5}), B = (2, -\sqrt{5})$, $C = (-2, -\sqrt{5})$, and $D = (-2, \sqrt{5})$. Note that these form a rectangle. Then we have

$$\begin{aligned}|z^2 - 4z + 9| + |z^2 + 4z + 9| &= |z - (2 + \sqrt{5}i)||z - (2 - \sqrt{5}i)| + |z - (-2 + \sqrt{5}i)||z - (-2 - \sqrt{5}i)| \\ &= |PA||PB| + |PC||PD| \\ &\geq 2(\text{Area}(\triangle PAB) + \text{Area}(\triangle PCD))\end{aligned}$$

since $\text{Area}(\triangle PAB) = \frac{1}{2}|PA||PB| \sin \angle APB \leq \frac{1}{2}|PA||PB|$ and $\text{Area}(\triangle PCD = \frac{1}{2}|PC||PD| \sin \angle CPD \leq \frac{1}{2}|PC||PD|$.

Finally, we have $2(\text{Area}(\triangle PAB) + \text{Area}(\triangle PCD)) \geq \text{Area}(ABCD) = 8\sqrt{5}$, where equality holds when P is inside the rectangle. This is true because $\triangle PAB$ and $\triangle PCD$ have the same base $AB = CD$ and the sum of the heights from P to AB and P to CD is equal to the length of the rectangle when P is inside the rectangle. When P is outside of the rectangle, the areas can only be larger.

Equality holds when $\angle APB = \angle CPD = 90°$, or when P is the intersection point of the circle whose diameter is AB and the circle whose diameter is CD. In the other words, equality holds when $z = \pm i$. So the minimum value of $|z^2 - 4z + 9| + |z^2 + 4z + 9|$ is $8\sqrt{5}$.

30. [**14**] Call an integer n *worthless* if the sum of its positive divisors (including n itself) is strictly less than $\frac{8n}{7}$. How many positive integers less than or equal to 1050 are worthless?

Proposed by: Sebastian Weinberger

Answer: $\boxed{237}$

Solution: Define a *worthwhile* integer as any integer that isn't worthless.

First, we note that any integer divisible by $2, 3, 5$, or 7 will be worthwhile as any integer n divisible by a prime p will have sum of divisors at least $\frac{n(p+1)}{p}$, but each of $\frac{2+1}{2}$, $\frac{3+1}{3}$, $\frac{5+1}{5}$, and $\frac{7+1}{7}$ are greater than or equal to $\frac{8}{7}$.

What other integers could be worthwhile? 1 is worthless. Also, for any prime $p > 7$, the sum of divisors will be $p + 1 < \frac{8p}{7}$ so those will be worthless. Since the next prime after 7 is 11 and $11^3 > 1050$, the only integers we have not accounted for yet are those of the form pq where p and q are primes. These will have divisors $1, p, q$, and pq in the case where $p \neq q$, and $1, p$, and p^2 in the case where $p = q$. In the latter case, we find that $1 + p + p^2 < \frac{8p^2}{7}$ always since $p \geq 11$. So the only remaining case is where $p \neq q$, in which case the sum of divisors is $1 + p + q + pq = (p + 1)(q + 1)$. We require that $\frac{(p+1)(q+1)}{pq} = \frac{p+1}{p}\frac{q+1}{q} \geq \frac{8}{7}$. Assume WLOG that $p < q$. Then $\frac{p+1}{p} > \frac{q+1}{q}$ so we require that $1 + \frac{1}{7} = \frac{8}{7} \leq \frac{p+1}{p}\frac{q+1}{q} < (\frac{p+1}{p})^2 = 1 + \frac{2}{p} + \frac{1}{p^2}$. Since p is a prime and $p > 7$, we quickly find that the only possible options for p are $p = 11$ and $p = 13$.

We can then proceed by inspection. When $p = 11$, we find that $q = 13, 17$, and $q = 19$ are the only q that will give rise to a worthwhile integer. When $p = 13$, we find that there is no q such that pq is worthwhile.

To summarize, our worthwhile integers are anything divisible by $2, 3, 5$, or 7, along with $11 \cdot$

$13, 11 \cdot 17$, and $11 \cdot 19$.

Conveniently, 1050 is divisible by each of $2, 3, 5$, and 7, so the number of integers not divisible by any of them between 1 and 1050 inclusive is just $1050 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} = 240$. Subtracting off the other three worthwhile integers, we find a total of 237 worthless integers.

31. [**15**] The Fibonacci sequence F_n is defined as $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 0$. Consider the polygon with vertices at Cartesian coordinates $(0,0), (F_0, F_1 + 1), (F_1, F_2 + 1), (F_2, F_3 + 1), \ldots, (F_{10}, F_{11} + 1)$ in order. What is the area of this polygon?

Proposed by: Joshua Kou

Answer: $\boxed{\dfrac{55}{2}}$

Solution: By shoelace we have

$$A = \frac{1}{2}\sum_{i=0}^{9}(F_{i+1}+1)F_{i+1} - F_i(F_{i+2}+1)$$

$$= \frac{1}{2}\left(\sum_{i=0}^{9}(F_{i+1}F_{i+1} - F_iF_{i+2}) + \sum_{i=0}^{9}(F_{i+1} - F_i)\right)$$

Using the identity $F_{i+1}F_{i+1} - F_iF_{i+2} = (-1)^i$, the first sum evaluates to 0. The second sum telescopes to $F_{10} - F_0$. Thus the answer is

$$A = \frac{1}{2}(F_{10}) = 27.5.$$

32. [**15**] Yor and Loid are playing a game with a set of n cards, numbered with the integers from 1 to n. Yor knows the value of n, but Loid does not.

Yor, with 2 of the cards already in her hand, tells Loid, "I know that the full deck consists of n cards. If you draw 2 cards, there is a $\frac{1}{30}$ chance that all four of our cards will sum to 45.

Loid draws two cards, and tells Yor, "Based on my cards, now I know there are two possible values for n."

Let Loid's highest card be s. What is the sum of all possible values of s?

Proposed by: Mason Yu

Answer: $\boxed{140}$

Solution: Suppose the sum of Yor's cards are k. Then Loid's cards need to sum up to $45 - k$ in order to make a sum of 45. Suppose there are m pairs left in the deck (not in Yor's hand) that sum up to $45 - k$. Then the probability of getting one of these pairs in 2 cards is $\frac{m}{\binom{n-2}{2}} = \frac{2m}{(n-2)(n-3)} = \frac{1}{30}$, with $2m \leq n$ (because the number of cards in pairs cannot exceed the number of cards total).

By inspection, the only (m, n) that can possibly work are $(10, 27), (7, 23)$, and $(4, 18)$. We want to eliminate exactly one of these possibilities. But first we have to check that the desired probability is attainable for some arbitrary pair of Yor's cards.

Case 1 : $(10, 27)$

Take the set of 12 pairs $(1, 24), (2, 23), \ldots, (12, 13)$ that add to 25. Then consider that Yor has drawn two cards that add to 20, each of which appears in different pairs in the set. Then we have 10 pairs left that sum to 25, which when added to Yor's cards is 45.

Case 2 : $(7, 23)$

Take the set of 9 pairs $(1, 18), (2, 17), \ldots, (9, 10)$ that add to 19. Then consider that Yor has drawn two cards that add to 26, each of which appears in different pairs in the set. Then we have 7 pairs left that sum to 19, which when added to Yor's cards is 45.

Case 3 : $(4, 18)$

Take the set of 4 pairs $(1, 9), (2, 8), (3, 7), (4, 6)$ that add to 10. Then consider that Yor has drawn two cards that add to 35, which would only be possible if she drew 17 and 18. Note that this is the only construction that works. The other set of 4 pairs that each add to 9 does not work because Yor cannot draw two cards that add to 36.

If we started with a set of 5 pairs that each add to 12, but then assumed that Yor had taken one or more of the cards from those pairs, she needs to take two cards that sum up to $45 - 12 = 33$, which is impossible because the highest sum she can take is $18 + 11 = 29$. (A set of 5 pairs that each add to 11 also fails in the same way.)

If we started with a set of 6 pairs that each add to 14 but then assumed that Yor had taken both cards from these pairs, she needs to take two cards that sum up to $45 - 14 = 31$, which is impossible because the highest sum she can take is $13 + 12 = 25$. (A set of 6 pairs that each add to 13 also fails in the same way.)

Therefore, the construction with 4 pairs and Yor drawing 17 and 18 is the only possible construction.

Now, to eliminate any case, Loid can draw a card higher than the n corresponding to that case (because then he knows there must be more than n cards.) This means that only case 3 must be eliminated, since it gets eliminated when any other case is eliminated. Additionally, to eliminate case 3, he can draw a 17 or 18, because he would then know that Yor would not have a 17 and 18, which means that case 3 would be impossible.

So Loid's highest card must be between 17 and 23 inclusive (any higher and case 2 would also be eliminated.) Therefore, the answer is $17 + 18 + 19 + 20 + 21 + 22 + 23 = 140$.

33. [**15**] Let $ABCDEFGHIJKL$ be a regular dodecagon with side length 1. Points W, X, Y, and Z are chosen uniformly at random from line segments AB, CD, EF, and GH respectively. What is the expected value of the area of quadrilateral $WXYZ$?

Proposed by: Joshua Kou

Answer: $\boxed{\dfrac{21\sqrt{3}}{16} + \dfrac{9}{4}}$

Solution: Suppose that instead of picking 4 points we picked 6 points, with the two additional points W' and X' on points on IJ and KL. Note that by symmetry across points WZ, the expected area of $WXYZW'X'$ will be twice the expected area of $WXYZ$.

The next observation that simplifies the computation is that we can extend the sides to meet at new points A', B', C', D', E', and F' which are at the intersections of KL, AB, CD, EF, GH, IJ, and KL respectively. We can compute the lengths of the new sides as the exterior angle is 30 degrees, giving extensions of length $1/\sqrt{3}$.

As a regular hexagon is composed of 6 equilateral triangles, we can compute the area of $A'B'C'D'E'F'$ as $7\sqrt{3}/12 + 1$. Then the expected area of $WXYZW'X'$ is $A'B'C'D'E'F' - 6E$[area of $A'WX'$] since the six missing triangles are indistinguishable.

Since the lengths of the sides are independent, E[area of $A'WX'$] is $\frac{1}{2}E$[length $A'W$]E[length WX'] $\sin 120°$. The minimum length of $A'W$ is the length of the extension $A'A = \frac{1}{\sqrt{3}}$. The maximum length is the length of $A'B = 1 + \frac{1}{\sqrt{3}}$. Thus the expected value of the length is their average, $\frac{1}{2} + \frac{1}{\sqrt{3}}$.

Thus the expected area of $WXYZ$ is with some arithmetic,

$$\frac{21\sqrt{3}}{16} + \frac{9}{4}$$

34. [**20**] Let $f(x)$ be a sixth-degree polynomial with no quadratic term satisfying $f(n) = n^{10}$ for $n \in 0, 1, 5, 6, 7, 8$. Estimate $\frac{f(100)}{10^{10}}$. If your estimate is E and the answer is A, you will receive $\max(0, \lfloor 20 \min(\frac{E}{A}, \frac{A}{E})^{0.6} \rfloor)$ points.

Proposed by: Sebastian Weinberger

Answer: $\boxed{5709923.5078}$

Solution: Since $f(x)$ has a root at zero, we define $g(x) = \frac{f(x)}{x} = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_0$ since $f(x)$ has no quadratic term. Now our constraints become $a_5n^5 + a_4n^4 + a_3n^3 + a_2n^2 + a_0 = n^9$ for $n \in \{1, 5, 6, 7, 8\}$. We can subtract the $n = 1$ case from the rest to obtain $a_5(n^5 - 1) + a_4(n^4 - 1) + a_3(n^3 - 1) + a_2(n^2 - 1) = n^9 - 1$ for $n \in \{5, 6, 7, 8\}$. Since this is an estimation problem, we can approximate $n^j - 1$ to n^j without too much loss of accuracy for $j \geq 2, n \geq 5$ so we might as well set $a_5n^5 + a_4n^4 + a_3n^3 + a_2n^2 = n^9$ or $a_5n^3 + a_4n^2 + a_3n + a_2 = n^7$ for $n \in \{5, 6, 7, 8\}$. We subtract the $n = 5$ case from the rest to obtain that $a_5(n^3 - 5^3) + a_4(n^2 - 5^2) + a_3(n - 5) = n^7 - 5^7$ for $n \in \{6, 7, 8\}$. This is a system of equations that is now within reach to solve, especially since we're allowed to be estimating some of the fractions. This gives us $a_5 \approx 65,000$ and $a_4 \approx -1,000,000$. So we can estimate that $\frac{f(100)}{10^{10}} \approx \frac{a_5 100^6 + a_4 100^5}{10^{10}} = 100a_5 + a_4 \approx 5,500,000$ which is indeed very close to the answer.

35. [**20**] We say a positive integer n is *sweet* if it satisfies both of the following properties:

- It has at least 6 divisors.
- Let $d_1 < d_2 < \ldots < d_k$ be the divisors of n. Then for all i such that $3 \leq i \leq k$, we have $d_i d_{i-2} \mid n$ or $n \mid d_i d_{i-2}$ (or both).

Estimate the number of sweet integers between 1 and 10^6 (inclusive). If your estimate is E and the answer is A, you will receive $\max\left(0, \left\lfloor 20 - \left(\frac{|A-E|}{200}\right)^{0.6} \right\rfloor\right)$ points.

Proposed by: Max Liu

Answer: $\boxed{41860}$

Solution: Listing out some examples leads us to the conjecture that all sweet numbers are of the form p^{k-1} for $k \geq 6$ or p^2q where $q > p^2$. We were unable to prove this conjecture, but we can verify that it holds for integers less than 10^6.

Most sweet integers will be of the form $2^2 \cdot q, 3^2 \cdot q$, and $5^2 \cdot q$. We estimate the number of primes less than $10^6/4, 10^6/9$, and $10^6/25$. The estimation $\pi(x) \approx \frac{x}{\ln x}$ yields an estimate of roughly 33452. Since this approximation undercounts the number of primes for small values of x, we multiply by a small factor (something between 1 and 1.5) to achieve a better estimate.

36. [**20**] Every year, thousands of new papers are published in mathematics — you are tasked with determining the relative number of published papers in the following categories: (A) Number Theory, (B) Probability Theory, (C) Dynamical Systems, (D) Complex Variables, (E) Geometric Topology, (F) Category Theory, and (G) Combinatorics. Based on 2024 data from arXiv.org, write down n of these categories (based on their corresponding letter), in ascending order from least to most papers published in 2024. If these n categories are in the correct relative order and $n \geq 4$, your team will earn $(n-2) \times (n-3)$ points. Otherwise, you will earn 0 points.

Proposed by: Amber Bajaj

Answer: F, D, E, C, A, B, G

Solution:
(A) Number Theory — 4463
(B) Probability Theory — 5987
(C) Dynamical Systems — 3984
(D) Complex Variables — 1436
(E) Geometric Topology — 2081
(F) Category Theory — 1105
(G) Combinatorics — 7599

Thus, the correct order of all 7 would be: F, D, E, C, A, B, G.