BrUMO

Team Round

Rules

For the team round, you will have 80 minutes to solve 15 short-answer questions. You are allowed to use writing utensils and scratch paper, but you may not use books, notes, calculators, slide rules, abaci, or any other computational aids. Similarly, you may not use graph paper, rulers, protractors, compasses, architectural tools, or any other drawing aids. You may collaborate with your teammates to solve these problems.

Scoring

For each problem your team solves correctly, your team will be awarded $e^{n/45} + \max(9 - \lfloor \ln 4N \rfloor, 4)$ points, where n is the problem number and N is the number of teams that solved the problem. There is no penalty for guessing.

Answer Format

Acceptable answers include integers, fractions, decimals, exponents, factorials, binomial coefficients, trigonometric functions, and radicals. You may also use any basic arithmetic operations, such as addition, subtraction, multiplication, and division, to express your answers. Summation or product notation will not be accepted.

Answers must be simplified in order to receive credit. Only exact answers will be accepted; in particular, decimal approximations will not receive credit. If a problem calls for a list (e.g. "Find all solutions x . . . "), you will be awarded full credit for a complete list in any order, but no partial credit for an incomplete list, as well as no credit for a list with extra elements.

Protests

If you have a protest, email brumo@brown.edu after the contest. All team round protests must be submitted through email by 2 pm EST.

Reminders

Please fill in your team name and ID on the answer sheet. Write your answers clearly in the corresponding box on the answer sheet. If you need scratch paper, please let your proctor know and we will bring it to you.

1. Find the smallest positive integer n such that n is divisible by exactly 25 different positive integers.

2. Two squares, $ABCD$ and $AEFG$, have equal side length x. They intersect at A and O. Given that $CO = 2$ and $OA = 2\sqrt{2}$, what is x?

3. Bruno and Brutus are running on a circular track with a 20 foot radius. Bruno completes 5 laps every hour, while Brutus completes 7 laps every hour. If they start at the same point but run in opposite directions, how far along the track's circumference (in feet) from the starting point are they when they meet for the sixth time? *Note: Do not count the moment they start running as a meeting point.*

4. What is the smallest positive integer n such that $z^n - 1$ and $(z - \sqrt{3})^n - 1$ share a common complex root?

5. Consider a pond with lily pads numbered from 1 to 12 arranged in a circle. Bruno the frog starts on lily pad 1. Each turn, Bruno has an equal probability of making one of three moves: jumping 4 lily pads clockwise, jumping 2 lily pads clockwise, or jumping 1 lily pad counterclockwise. What is the expected number of turns for Bruno to return to lily pad 1 for the first time?

6. 4 bears — Aruno, Bruno, Cruno and Druno — are each given a card with a positive integer and are told that the sum of their 4 numbers is 17. They cannot show each other their cards, but discuss a series of observations in the following order:

   Aruno: "I think it is possible that the other three bears all have the same card."
   Bruno: "At first, I thought it was possible for the other three bears to have the same card. Now I know it is impossible for them to have the same card."
   Cruno: "I think it is still possible that the other three bears have the same card."
   Druno: "I now know what card everyone has."

   What is the product of their four card values?

7. Digits 1 through 9 are placed on a 3x3 square such that all rows and columns sum to the same value. Please note that diagonals do not need to sum to the same value. How many ways can this be done?

8. Define the operation $\oplus$ by
$$x \oplus y = xy - 2x - 2y + 6.$$
   Compute all complex numbers a such that
$$a \oplus (a \oplus (a \oplus a)) = a.$$

9. Define the function f on positive integers
$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$$
   Let $S(n)$ equal the smallest positive integer k such that $f^k(n) = 1$. How many positive integers satisfy $S(n) = 11$?

10. Let $ABCDEF$ be a convex cyclic hexagon. Suppose that $AB = DE = \sqrt{5}, BC = EF = 3$, and $CD = FA = \sqrt{20}$. Compute the circumradius of $ABCDEF$.

11. A repetend is the infinitely repeated digit sequence of a repeating decimal. What are the last three digits of the repetend of the decimal representation of $\frac{1}{727}$, given that the repetend has a length of 726? Express the answer as a three-digit number. Include preceding zeros if there are any.

12. Consider a 54-deck of cards, i.e. a standard 52-card deck together with two jokers. Ada draws cards from the deck until Ada has drawn an ace, a king, and a queen. How many cards does Ada pick up on average?

13. Let ω be a circle, and let a line ℓ intersect ω at two points, P and Q. Circles ω_1 and ω_2 are internally tangent to ω at points X and Y, respectively, and both are tangent to ℓ at a common point D. Similarly, circles ω_3 and ω_4 are externally tangent to ω at X and Y, respectively, and are tangent to ℓ at points E and F, respectively.

    Given that the radius of ω is 13, the segment $\overline{PQ} = 24$, and $\overline{YD} = \overline{YE}$, find the length of segment $\overline{YF}$.

14. Let f be a degree 7 polynomial satisfying

$$f(k) = \frac{1}{k^2}$$

    for $k \in \{1 \cdot 2, 2 \cdot 3, \ldots, 8 \cdot 9\}$. Find $f(90) - \frac{1}{90^2}$.

15. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let D be a point on the circumcircle of $\triangle ABC$ on minor arc AB. Let $\overline{AD}$ intersect the extension of $\overline{BC}$ at E. Let F be the midpoint of segment AC, and let G be the intersection of $\overline{EF}$ and $\overline{AB}$. Let the extension of $\overline{DG}$ intersect $\overline{AC}$ and the circumcircle of $\triangle ABC$ at H and I, respectively. Given that $DG = 3, GH = 5$, and $HI = 1$, compute the length of AE.