1. Find the smallest positive integer n such that n is divisible by exactly 25 different positive integers.

   Proposed by: Adeethyia Shankar

   Answer: $\boxed{1296}$

   Solution: Since $25 = (24+1)\times(0+1) = (4+1)\times(4+1)$, $n = p^{24}$ or $n = p^4 \times q^4$, for distinct prime numbers p and q. To obtain the smallest positive integer n, we choose $p = 2$ and $q = 3 \Rightarrow n = 2^4 \times 3^4 = 1296$.

2. Two squares, $ABCD$ and $AEFG$, have equal side length x. They intersect at A and O. Given that $CO = 2$ and $OA = 2\sqrt{2}$, what is x?

   Proposed by: Jaeho Lee

   Answer: $\boxed{1+\sqrt{3}}$

   Solution: We know that $DO = CD - CO = x - 2$. Using the Pythagorean theorem on $\triangle AOD$, we have

$$\begin{aligned} DO^2 + AD^2 &= OA^2 \\ (x-2)^2 + x^2 &= 8 \\ 2x^2 - 4x + 4 &= 8 \\ x^2 - 2x - 2 &= 0. \end{aligned}$$

   Using the quadratic formula, we find that $x = 1 \pm \sqrt{3}$. Since x must be positive, we have $x = 1 + \sqrt{3}$.

3. Bruno and Brutus are running on a circular track with a 20 foot radius. Bruno completes 5 laps every hour, while Brutus completes 7 laps every hour. If they start at the same point but run in opposite directions, how far along the track's circumference (in feet) from the starting point are they when they meet for the sixth time? *Note: Do not count the moment they start running as a meeting point.*

   Proposed by: Annabel Qin

   Answer: $\boxed{20\pi}$

   Solution: Since Bruno and Brutus are running in opposite directions, the speed at which they run towards each other is 12 laps per hour. This means that it takes them half an hour to cross paths for the sixth time. After half an hour, Bruno has completed 2.5 laps and Brutus has completed 3.5 laps. They are both on the opposite side of the track at this point, so they are halfway along the circumference of the track. The circumference is 40π, so there are 20π feet away from the starting point.

4. What is the smallest positive integer n such that $z^n - 1$ and $(z - \sqrt{3})^n - 1$ share a common complex root?

   Proposed by: Ethan Bove

   Answer: $\boxed{12}$

   Solution: Observe that the roots of $z^n - 1$ lie on the circle $|z| = 1$ in the complex plane. Similarly, the roots of $(z-\sqrt{3})^n - 1$ lie on the circle $|z - \sqrt{3}| = 1$. These circles intersect at $\frac{\sqrt{3}}{2} + \frac{1}{2}i$ and $\frac{\sqrt{3}}{2} - \frac{1}{2}i$. These are equal to $e^{\frac{\pi}{6}i}$ and $e^{-\frac{\pi}{6}i}$, respectively, so we need $e^{\frac{n\pi}{6}i} - 1 = 0$. The smallest n for which this is true is $n = 12$, and we can check that $e^{\frac{\pi}{6}i}$ is a root of both polynomials.

5. Consider a pond with lily pads numbered from 1 to 12 arranged in a circle. Bruno the frog starts on lily pad 1. Each turn, Bruno has an equal probability of making one of three moves: jumping 4 lily pads clockwise, jumping 2 lily pads clockwise, or jumping 1 lily pad counterclockwise. What is the expected number of turns for Bruno to return to lily pad 1 for the first time?

   Proposed by: Annabel Qin

   Answer: $\boxed{12}$

   Solution: We define E_k as the expected number of turns for Bruno to return to lily pad 1 starting from lily pad k. The goal is to solve for E_1. The recurrence relation yields the following system of equations:

$$
\begin{aligned}
E_1 &= 1 + \frac{1}{3}E_5 + \frac{1}{3}E_3 + \frac{1}{3}E_{12}, \\
E_2 &= 1 + \frac{1}{3}E_6 + \frac{1}{3}E_4, \\
E_3 &= 1 + \frac{1}{3}E_7 + \frac{1}{3}E_5 + \frac{1}{3}E_2, \\
&\vdots \\
E_{11} &= 1 + \frac{1}{3}E_3 + \frac{1}{3}E_{10}, \\
E_{12} &= 1 + \frac{1}{3}E_4 + \frac{1}{3}E_2 + \frac{1}{3}E_{11}.
\end{aligned}
$$

   Note that we can assume there is a finite value for any E_k since we will reach lily pad 1 from any of the other 11 in finite time. Therefore, we are able to sum up all the equations and cancel terms on both sides:

$$\sum_{k=1}^{12} E_k = \sum_{k=2}^{12} E_k + 12.$$

   Thus, $E_1 = 12$.

6. 4 bears — Aruno, Bruno, Cruno and Druno — are each given a card with a positive integer and are told that the sum of their 4 numbers is 17. They cannot show each other their cards, but discuss a series of observations in the following order:

   Aruno: "I think it is possible that the other three bears all have the same card."
   Bruno: "At first, I thought it was possible for the other three bears to have the same card. Now I know it is impossible for them to have the same card."
   Cruno: "I think it is still possible that the other three bears have the same card."
   Druno: "I now know what card everyone has."

   What is the product of their four card values?

   Proposed by: Amber Bajaj

   Answer: $\boxed{160}$

   Solution: Let Aruno, Bruno, Cruno, and Druno have cards with values a, b, c, and d, respectively. The first observation tells us that $17 - a$ is divisible by 3, so a can be $2, 5, 8, 11$, or 14.
   Similarly, the second observation tells us that b can be $2, 5, 8, 11$, or 14. However, knowing that $3a \in \{6, 15, 24, 33, 42\}$, Bruno concludes that a, c, and d cannot be the same. This means that b cannot be equal to $17 - 6, 17 - 15, 17 - 24, 17 - 33$, or $17 - 42$. In particular, b cannot be 2 or 11, so we know that b is $5, 8$, or 14.

The third observation also tells us that c can be $2, 5, 8, 11$, or 14. Cruno knows that $3b \in \{15, 24, 42\}$. Since 24 and 42 are too large, Cruno must believe it is possible that $3b = 15$, so we must have $c = 2$.
Finally, Druno can figure out a and b. With the constraint that $c = 2$, the possible values of $a + b$ are $7, 10$, or 13. The only value of $a + b$ that determines a and b is 7, which determines that $a = 2$ and $b = 5$. This gives us $d = 8$, so $abcd = 160$.

7. Digits 1 through 9 are placed on a 3x3 square such that all rows and columns sum to the same value. Please note that diagonals do not need to sum to the same value. How many ways can this be done?

   Proposed by: Ethan Bove

   Answer: $\boxed{72}$

   Solution: There will be a unique solution up to permuting the columns and flipping along the y axis. Since $1 + \cdots + 9 = 45$, it follows that each row and column must sum to 15. By permuting the rows and columns in a correct configuration, we can assume that 9 is in the top left. The only two ways to make 15 as a sum including 9 are as $9 + 5 + 1$ and $9 + 4 + 2$. Thus, these must be the content of the row and column containing 9 in some order. By rotating and permuting rows/columns as necessary, we can assume our solution is in the form:

$$\begin{bmatrix} 9 & 4 & 2 \\ 5 & ? & ? \\ 1 & ? & ? \end{bmatrix}$$

   The bottom row must contain 8 and 6, as this is the only way to sum with 1 to fifteen using digits not currently used. They must go in the order $1, 8, 6$, as the other permutation would require 5 to appear a second time in the center square. From here, it follows that the only solution will be some variation of:

$$\begin{bmatrix} 9 & 4 & 2 \\ 5 & 3 & 7 \\ 1 & 8 & 6 \end{bmatrix}$$

   Thus, the problem reduced to counting the number of symmetries of this square. There will be 3! ways to permute the columns, 3! ways to permute the rows, along with an additional $3! \cdot 3!$ solutions formed by switching the axes. There are thus $3! \cdot 3! \cdot 2 = 72$ solutions.

8. Define the operation $\oplus$ by

$$x \oplus y = xy - 2x - 2y + 6.$$

   Compute all complex numbers a such that

$$a \oplus (a \oplus (a \oplus a)) = a.$$

   Proposed by: Aidan Hennessey

   Answer: $\boxed{2, 3, \frac{3 + i\sqrt{3}}{2}, \frac{3 - i\sqrt{3}}{2}}$

   Solution: Factor as $x \oplus y = xy - 2x - 2x + 6 = (x - 2)(y - 2) + 2$. Substitute $b = a - 2$. Then, $a \oplus a = b^2 + 2$. Plugging this in, $(a \oplus (a \oplus a)) = a \oplus (b^2 + 2) = b^3 + 2$. Plugging in one last time, $a \oplus (a \oplus (a \oplus a)) = a \oplus (b^3 + 2) = b^4 + 2$. Recalling the original equation, we now wish to solve $b^4 + 2 = b + 2$. Moving everything to one side and factoring gives $b(b - 1)(b^2 + b + 1) = 0$.

This yields $b = 0, 1, \frac{-1+i\sqrt{3}}{2}$, and $\frac{-1-i\sqrt{3}}{2}$. Since $a = b + 2$, the solutions for a are $2, 3, \frac{3+i\sqrt{3}}{2}$, and $\frac{3-i\sqrt{3}}{2}$.

9. Define the function f on positive integers

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ n+1 & \text{if } n \text{ is odd} \end{cases}$$

Let $S(n)$ equal the smallest positive integer k such that $f^k(n) = 1$. How many positive integers satisfy $S(n) = 11$?

Proposed by: Joshua Kou

Answer: $\boxed{89}$

Solution: Let a_k be the number of even positive integers n not equal to 2 satisfying $S(n) = k$ and b_k be the number of odd positive integers n not equal to 1 satisfying $S(n) = k$.

Suppose that $S(n) = k + 1$ for some $k \geq 2$. If $n \neq 2$ is even, then $S(\frac{n}{2}) = k$ and $\frac{n}{2} \neq 1$. Since both even and odd integers can be written as $\frac{n}{2}$, we have $a_{k+1} = a_k + b_k$. If $n \neq 1$ is odd, then $S(n + 1) = k$ and $n + 1 \neq 2$. Since only even integers can be written as $n + 1$ when n is odd, we have $b_{k+1} = a_k$.

Let $c_k = a_k + b_k$. Then

$$\begin{aligned} c_{k+1} &= a_{k+1} + b_{k+1} = (a_k + b_k) + a_k \\ &= (a_k + b_k) + (a_{k-1} + b_{k-1}) = c_k + c_{k-1} \end{aligned}$$

for $k \geq 3$.

We can manually compute $c_2 = 1$ and $c_3 = 2$, and applying the recursive formula yields $c_{11} = 89$.

10. Let $ABCDEF$ be a convex cyclic hexagon. Suppose that $AB = DE = \sqrt{5}, BC = EF = 3$, and $CD = FA = \sqrt{20}$. Compute the circumradius of $ABCDEF$.

Proposed by: Max Liu

Answer: $\boxed{\dfrac{1+\sqrt{31}}{2}}$

Solution: By symmetry, AD is a diameter. Let the circumradius be R. Let $AB = a, BC = b$, and $CD = c$. Then $AC^2 = 4R^2 - c^2$ and $BD^2 = 4R^2 - b^2$. By Ptolemy's theorem,

$$(4R^2 - c^2)(4R^2 - b^2) = (2aR \cdot R + b \cdot d)^2.$$

Simplifying this yields

$$4R^3 - (a^2 + b^2 + c^2)R^2 - abc = 0.$$

We plug in $a = \sqrt{5}, b = 3$, and $c = \sqrt{20}$. This yields

$$4R^3 - 34R - 30 = 0.$$

We find that $R = -1$ is a root, so we can factor this as $(R + 1)(4R^2 - 4R - 30)$. The quadratic yields $\frac{1+\sqrt{31}}{2}$ as the only positive solution.

11. A repetend is the infinitely repeated digit sequence of a repeating decimal. What are the last three digits of the repetend of the decimal representation of $\frac{1}{727}$, given that the repetend has a length of 726? Express the answer as a three-digit number. Include preceding zeros if there are any.

    Proposed by: Haoyang Xu

    Answer: $\boxed{337}$

    Solution: Let the repetend of $\frac{1}{727}$ be R. Since R has 726 digits, we know that

$$\frac{1}{727} = \frac{R}{10^{726}} + \frac{R}{10^{2\cdot 726}} + \cdots = \frac{R}{10^{726}-1}.$$

    This yields $727R = 10^{726} - 1$. Let the last three digits be a, b, and c. We solve $100a + 10b + c \equiv 999 \pmod{1000}$.

    Looking at the units digit, we have $7c \equiv 9 \pmod{10}$, so $c = 7$. Then, considering the tens digit, we have $7 \times 10b + 27 \times 7 \equiv 99 \pmod{100}$, which yields $b = 3$. Finally, for the hundreds digit, we have $7 \times 100a + 27 \times 30 + 727 \times 7 \equiv 999 \pmod{1000}$, which yields $a = 3$. The last three digits are 337.

12. Consider a 54-deck of cards, i.e. a standard 52-card deck together with two jokers. Ada draws cards from the deck until Ada has drawn an ace, a king, and a queen. How many cards does Ada pick up on average?

    Proposed by: Joshua Kou

    Answer: $\boxed{\frac{737}{39}}$

    Solution: Let X, Y, Z be the positions of the first ace, king, queen (where the deck is 1-indexed). We wish to compute $E[\max(X, Y, Z)] = E[X + Y + Z - \min(X, Y) - \min(Y, Z) - \min(X, Z) + \min(X, Y, Z)]$. By linearity of expectation, we can split this sum. Consider the 4 Aces as dividers, which divide the remaining 50 cards in the deck into 5 sections. Then the expected number of cards upon drawing the first Ace, $E[X] = 1 + 50/5 = 11 = E[Y] = E[Z]$ by symmetry. Note that $\min(X, Y)$ is simply the first instance of a Ace or King, so we can effectively treat the Aces and Kings as indistinguishable and apply the divider idea once again, giving $E[\min(X, Y)] = 1 + 46/9 = 55/9$. Similarly, $E[\min(X, Y, Z)] = 1 + 42/13 = 55/13$. Combining, we have

$$E[\max(X, Y, Z)] = 3 \cdot 11 - 3 \cdot 55/9 + 55/13 = 737/39$$

13. Let ω be a circle, and let a line ℓ intersect ω at two points, P and Q. Circles ω_1 and ω_2 are internally tangent to ω at points X and Y, respectively, and both are tangent to ℓ at a common point D. Similarly, circles ω_3 and ω_4 are externally tangent to ω at X and Y, respectively, and are tangent to ℓ at points E and F, respectively.

    Given that the radius of ω is 13, the segment $\overline{PQ} = 24$, and $\overline{YD} = \overline{YE}$, find the length of segment $\overline{YF}$.

    Proposed by: Joey Chun

    Answer: $\boxed{5\sqrt{2}}$

Solution: Let O be the center of ω. Draw a line perpendicular to ℓ that passes through O. This line intersects ω at two points, which we denote as M (closer to Y) and M' (closer to X).

We first show that the lines XD and YF intersect ω at M. The homothety centered at X that sends ω_1 to ω also sends D to M. Therefore, X, D, and M are collinear. Similarly, the homothety centered at Y that sends ω_4 to ω also sends F to M, so Y, F, and M are collinear. This proves the claim. (Alternatively, those who know the Death Star lemma can directly apply it.)

Using a similar argument, we find that lines YD and XE intersect ω at M'. Since MM' is a diameter of ω, we know that $\angle MXM' = 90°$, so $\angle MXE = \angle DXE = 90°$. Similarly, $\angle M'YM = 90°$, so $\angle M'YF = \angle DYF = 90°$.

Let H be the intersection of MM' and PQ. Apply the radical axis theorem on the circumcircle of $DHM'X$, the circumcircle of $DHMY$, and ω: the lines $M'X$, MY, and ℓ must meet at one point, which implies that $E = F$ and they lie on ℓ.

Since $\overline{YD} = \overline{YE}$, it follows that $\triangle YDE$ is a right isosceles triangle, as well as $\triangle HDM'$ and $\triangle HEM$ by similarity. Also, notice that $OH = \sqrt{13^2 - (\frac{24}{2})^2} = 5$ by Pythagorean Theorem on $\triangle OHQ$. Thus,

$$DE = HE - HD = HM - HM' = (R + OH) - (R - OH) = 2OH = 10.$$

Therefore,

$$YF = YE = \frac{10}{\sqrt{2}} = 5\sqrt{2}.$$

14. Let f be a degree 7 polynomial satisfying

$$f(k) = \frac{1}{k^2}$$

for $k \in \{1 \cdot 2, 2 \cdot 3, \ldots, 8 \cdot 9\}$. Find $f(90) - \frac{1}{90^2}$.

Proposed by: Joshua Kou

Answer: $\boxed{-\frac{2431}{50}}$

Solution: Define the ninth-degree polynomial $g(x) = x^2 f(x) - 1$. Then g has roots at each of $\{1 \cdot 2, 2 \cdot 3 ..., 8 \cdot 9\}$, and in particular $g(x) = h(x)(x - 1 \cdot 2)(x - 2 \cdot 3)...(x - 8 \cdot 9)$ where $h(x)$ is linear. Let

$$h(x) = c(x + b)$$

$$g(x) = c(x + b)(x - 1 \cdot 2)(x - 2 \cdot 3)...(x - 8 \cdot 9)$$

Then since $x^2 \mid g(x) + 1$, the coefficient of x in $g(x) + 1$ must be 0. Let S be the product of the roots. Then we know that

$$0 = \sum \frac{S}{r_i}$$

over the roots r_i. Thus

$$0 = \sum \frac{S}{r_i} = -\frac{S}{b} + \sum_{i=1}^{8} \frac{S}{i(i+1)}$$

$$\frac{1}{b} = \sum_{i=1}^{8} \frac{1}{i(i+1)} = 1 - \frac{1}{9} = \frac{8}{9}$$

Thus

$$b = \frac{9}{8}$$

Furthermore, since $x^2 \mid g(x)+1$, the constant term in $g(x)+1$, i.e. $g(0)+1$ must be 0.

$$c\frac{9}{8}\prod_{i=1}^{8} -i(i+1) = -1$$

$$c = -\frac{8}{9! \cdot 9!}$$

Thus

$$g(x) = x^2 f(x) - 1 = -\frac{8}{9!^2}\left(x + \frac{9}{8}\right)(x - 1\cdot 2)(x - 2\cdot 3)...(x - 8\cdot 9)$$

Plugging in $x = 9 \cdot 10$,

$$90^2 f(90) - 1 = -\frac{8}{9!^2}\left(9\cdot 10 + \frac{9}{8}\right)(9\cdot 10 - 1\cdot 2)(9\cdot 10 - 2\cdot 3)...(9\cdot 10 - 8\cdot 9)$$

$$= -\frac{8}{9!^2}\cdot\frac{9^3}{8}(8\cdot 11)(7\cdot 12)\ldots(1\cdot 18) = -\frac{8}{9!^2}\cdot\frac{9^3}{8}\cdot\frac{8!18!}{10!}$$

So

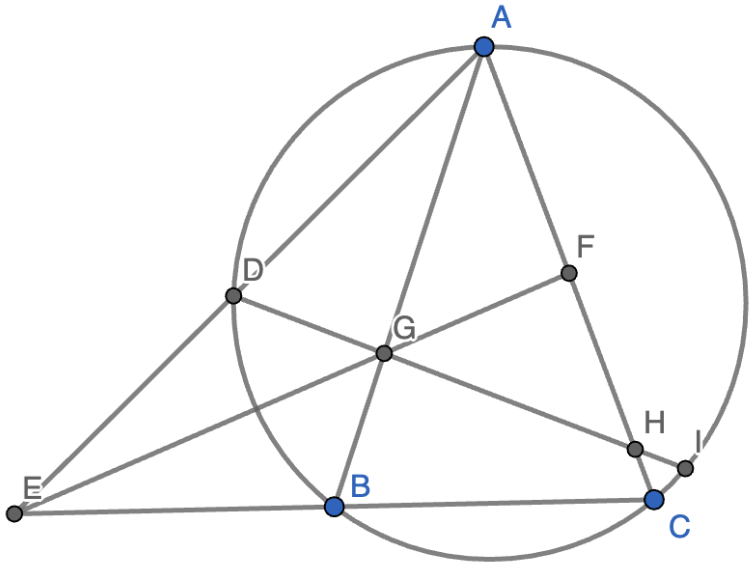
$$f(90) - \frac{1}{90^2} = -\frac{1}{9^2 10^2}\cdot\frac{8}{9!^2}\cdot\frac{9^3}{8}\cdot\frac{8!18!}{10!} = -\frac{18!}{10^2\cdot 9!\cdot 10!} = -\frac{2431}{50}.$$

15. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$. Let D be a point on the circumcircle of $\triangle ABC$ on minor arc AB. Let $\overline{AD}$ intersect the extension of $\overline{BC}$ at E. Let F be the midpoint of segment AC, and let G be the intersection of $\overline{EF}$ and $\overline{AB}$. Let the extension of $\overline{DG}$ intersect $\overline{AC}$ and the circumcircle of $\triangle ABC$ at H and I, respectively. Given that $DG = 3, GH = 5$, and $HI = 1$, compute the length of AE.

    Proposed by: Max Liu

    Answer: $\boxed{\frac{9\sqrt{30}}{4}}$

    Solution:



First, note that $\angle ADB = 180^\circ - \angle C = 180^\circ - \angle B = \angle ABE$, so $\triangle ADB \sim \triangle ABE$ and therefore $\overline{AB}$ is tangent to the circumcircle of $\triangle BDE$. Let ω be this circle. Let G' be the intersection of $\overline{AB}$ and the line through D tangent to ω. Since $\overline{G'B}$ and $\overline{G'D}$ are tangent to the circumcircle of $\triangle BDE$, it follows that $\overline{EG'}$ is the E-symmedian of $\triangle BDE$.

We show that $\overline{EG'}$ intersects $\overline{AC}$ at F. Reflect B and D over the angle bisector of $\angle BED$ to get B' and D'. Then $\overline{B'D'} \parallel \overline{AC}$ and $\overline{EG'}$ coincides with the E-median of $\triangle BDE$. Therefore, $\overline{E'G}$ coincides with the E-median of $\triangle CAE$, so $\overline{E'G}$ intersects $\overline{AC}$ at F. It follows that $G' = G$.

The fact that $\overline{DG}$ is tangent to ω gives us $DG = BG$, so $AG = IG$.

Since $\overline{DG}$ is tangent to ω, we have

$$\begin{aligned} \angle ADG &= 180^\circ - \angle DEG \\ &= \angle DBE \text{ (since } \overline{DG} \text{ is tangent to } \omega) \\ &= 180 - \angle DBC \\ &= \angle A. \end{aligned}$$

Therefore, $\triangle AHD$ is isosceles and $AH = DH$.

We know $AH = 8, GH = 5, DG = 3$, and $AG = 6$, so we can compute $AD = \sqrt{\frac{216}{5}}$ using Stewart's formula. Since $DG = BG$ and $AG = IG$, we know that $AB = DI = 9$. By power of a point, we have $AE = AB^2/AD = \frac{9\sqrt{30}}{4}$.