NYCMT 2024-2025 Homework #6 Solutions

NYCMT

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Problem 1. Show that for all positive integers n,

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor = \left\lfloor \sqrt{4n+3} \right\rfloor.$$

Solution. Let $a = \left\lfloor \sqrt{4n} \right\rfloor$ and $b = 4n - \left\lfloor \sqrt{4n} \right\rfloor^2$, so that we can write

$$n = \frac{1}{4}\left(a^2 + b\right)$$

so that a is maximized. Note that $a^2 + b$ must be a multiple of 4, since n is an integer. First, we want to find the value of

$$\left\lfloor \sqrt{a^2 + b + 1} \right\rfloor.$$

Since a is maximal, $\left\lfloor \sqrt{a^2 + b + 1} \right\rfloor$ must either be a or $a + 1$. For it to equal $a + 1$, it must be the case that $b = 2a$ (because if b were any bigger, a would not be maximized). Also notice that a must be even for this to be the case, since we know that $4 \mid a^2 + b$. Thus, we have that

$$\lfloor 4n+1 \rfloor = \begin{cases} a & 0 \le b < 2a \\ a+1 & b = 2a \end{cases}$$

Since $4n + 2$ and $4n + 3$ are not congruent to 0 or 1 mod 4, they cannot be perfect squares, so it must be the case that

$$\left\lfloor \sqrt{4n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor = \left\lfloor \sqrt{4n+3} \right\rfloor.$$

Now, we want to show that

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor.$$

We claim that

$$\sqrt{n} + \sqrt{n+1} > \sqrt{4n+1}$$

for all positive n. Since $n > 0$, we have that $n^2 + n > n^2$, or $n(n+1) > n^2$. Since both sides are positive and $\sqrt{x}$ is an increasing function, we can take the square root of both sides to get that

$$\sqrt{n(n+1)} > n.$$

We can then multiply by 2 and add $2n + 1$ on both sides to get

$$n + n + 1 + 2\sqrt{n(n+1)} > 4n + 1,$$

which collapses into

$$\left(\sqrt{n} + \sqrt{n+1}\right)^2 > \left(\sqrt{4n+1}\right)^2.$$

Again, since both sides are positive, we can take the square root of both sides to obtain the desired result. Because of this, it suffices to show that

$$\sqrt{n} + \sqrt{n+1} < 1 + \left\lfloor \sqrt{4n+1} \right\rfloor$$

If $0 \leq b < 2a$, this is equivalent to showing that

$$\sqrt{\frac{1}{4}\left(a^2 + b\right)} + \sqrt{\frac{1}{4}\left(a^2 + b\right) + 1} = \frac{1}{2}\left(\sqrt{a^2 + b} + \sqrt{a^2 + b + 4}\right) < a + 1.$$

Since $a^2 + b$ must be divisible by 4, if a is odd, then the largest possible value of b is $2a - 3$, and the inequality is clearly true for $0 \leq b \leq 2a - 3$. If a is even, then the largest possible value of b is $2a - 4$, so the inequality will also hold true.

If $b = 2a$, then we just want to show that

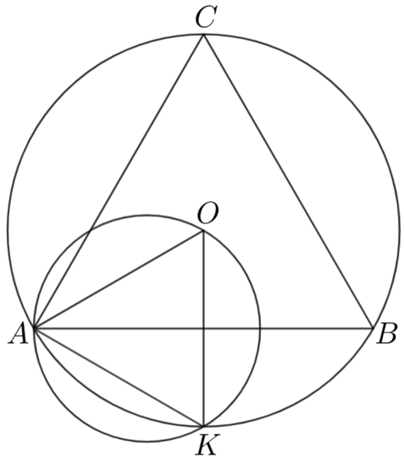
$$\frac{1}{2}\left(\sqrt{a^2 + 2a} + \sqrt{a^2 + 2a + 4}\right) < a + 2.$$

Since $a \geq 0$, we have that $a^2 + 2a + 4 \leq a + 4a + 4 = (a+2)^2$, which implies the result. $\square$

Problem 2. A finite number of points are marked on the plane, no three of them collinear. A circle is circumscribed around each triangle with marked vertices. Is it possible that centers of all of these circles are also marked?

Answer. No

Solution. Suppose that such a configuration exists. Then consider the three points that make up the triangle with the smallest circumradius, and call them A, B, and C. Let O be the circumcenter of $\triangle ABC$, and R be the circumradius. By our assumption, O must be a marked point. WLOG let $\angle A \leq \angle B \leq \angle C$.



Define a to be the length of the side of $\triangle ABC$ opposite $\angle A$ (the length of side BC), $b = CA$, and $c = AB$. By the Extended Law of Sines,

$$a = 2R\sin\angle A,$$

and the lengths of b and c follow a similar pattern. Now let ω_A be the circumcircle of $\triangle OBC$, ω_B be the circumcircle of $\triangle OCA$ and ω_C to be the circumcircle of $\triangle OAB$. Then if R_A is the radius of ω_A, by the Extended Law of Sines,

$$R_A = \frac{BC}{2\sin\angle BOC} = \frac{2R\sin\angle A}{2} \cdot \frac{1}{\sin 2\angle A} = \frac{R\sin\angle A}{2\sin\angle A\cos\angle A} = \frac{R}{2\cos\angle A}.$$

Similarly, if R_B is the radius of ω_B and R_C is the radius of ω_C, then

$$R_B = \frac{R}{2\cos\angle B} \text{ and } R_C = \frac{R}{2\cos\angle C}.$$

Then since we have assumed that (ABC) has the smallest circumradius,

$$R \geq R_A, R_B, R_C,$$

so $R \geq \frac{R}{2\cos\angle A}$, which means that $\cos\angle A \geq \frac{1}{2}$. Similarly, $\cos B, \cos C \geq \frac{1}{2}$. Since $\cos x$ is decreasing on $x \in (0, \pi)$, this implies that

$$\angle A, \angle B, \angle C \leq 60^\circ,$$

but since angles of a triangle must add up to 180°, this implies that $\angle A = \angle B = \angle C = 60^\circ$, so $\triangle ABC$ is an equilateral triangle.

Now, let K be the circumcenter of $\triangle OAB$, and note that it must be a marked point. Notice that K is the reflection of O over AB, so $\triangle OAK$ is an equilateral triangle with side length OA. However,

$$OA = \frac{AB}{\sqrt{3}} < AB,$$

and since they are both equilateral triangles, this means that the circumradius of $\triangle OAK$ is less than the circumradius of $\triangle ABC$. We have thus reached a contradiction. $\square$

Solution. Suppose that such a configuration exists. Then consider the triangle formed by two adjacent points A an B on the convex hull of the set of points, and another point P_1 in the set. For $i > 1$, define P_i to be the circumcenter of $\triangle AP_{i-1}B$. Since we are assuming that the points satisfy the given condition, given any P_i, P_{i+1} must lie in the set as well.

Let $\theta_i = \angle AP_iB$. Then due to the Inscribed Angle Theorem, if $\theta_i < 90°$, then $\theta_{i+1} = 2\theta_i$. This means that eventually, there must exist some θ_k such that $\theta_k \geq 90°$ (and $\theta_i < 90°$ for all $1 \leq i < k$).

If $\theta_k = 90°$, then P_{k+1} must be the midpoint of AB. However, this contradicts the assumption that A and B are adjacent. Similarly, if $\theta_k > 90°$, then P_{k+1} must lie outside the convex hull of the set of points, another contradiction. $\square$

Problem 3. The recursive integer sequence is defined by $F_n = \frac{F_{n-1}-1}{2}$, with $F_1 = n$ (notice this sequence terminates when there are no longer integer values). If there exists an integer $i > 1$ such that $F_i \mid n$, must it be true that n is of the form $2^k - 1$ for integer k?

Solution. Notice that the sequence must terminate. For all integer $F_k \geq 1$, $F_{k+1} = \frac{F_{k-1}-1}{2} \geq \frac{1-1}{2} = 0$, and $F_{k-1} = 2F_k + 1 > F_k$, so the recursive sequence is a decreasing sequence of positive integers. Hence, by infinite descent, the sequence must terminate.

Consider the inverse of the linear recurrence. If there exists another integer $i > 1$ such that $F_i \mid F_1$, then let $G_1, G_2, G_3, \ldots$ be another recursive sequence defined by $G_1 = F_i$ and $G_n = 2G_{n-1} + 1$. $F_i \mid F_1$ is then analogous to $G_1 \mid G_i$, as the sequence is simply reversed and the terminating condition is removed.

We can then find the closed form for G_n. Notice that if we express G_n in binary, the recursive condition is equivalent to appending a 1 to the end of the binary representation of G_{n-1}. Thus, G_n in binary is the binary representation of G_1 with $n-1$ ones appended to the end, so

$$G_n = 2^{n-1}G_1 + \sum_{i=0}^{n-2} 2^i = 2^{n-1}G_1 + 2^{n-1} - 1.$$

Thus, $G_1 \mid G_i$ implies that

$$G_1 \mid 2^{i-1}G_1 + 2^{i-1} - 1,$$

which can only happen if $G_1 \mid 2^{i-1} - 1$. Analyzing the final condition, if G_i is of the form $2^k - 1$, then

$$G_i = 2^{i-1}G_1 + 2^{i-1} - 1 = 2^k - 1,$$

which means that

$$G_1 = 2^{k+1-i} - 1.$$

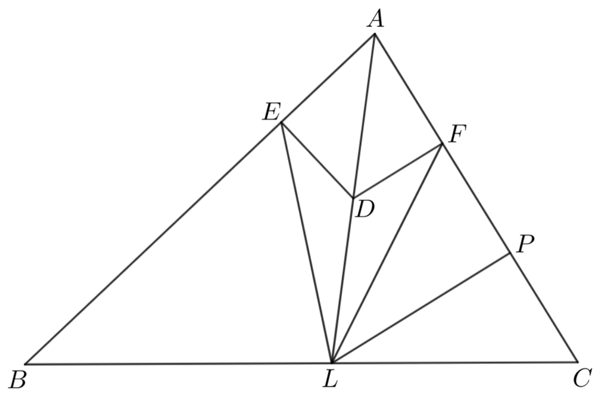
Thus, as long as G_1 is not one less than a power of two, then $G_i = n$ is no longer of that form too. Combined with the condition that $G_1 \mid 2^{i-1} - 1$, the question is simply asking if there exists a divisor of a number one less than a power of two such that it itself is not one less than a power of two.

To find an example that disproves the problem statement, consider the equation $2^4 - 1 = 15$, and note that 5 is a divisor that is not one less than a power of 2. Then, $G_1 = 5$ and $i = 5$, so

$$G_5 = 2^4(5) + 2^4 - 1 = 95.$$

Hence, $n = 95$ is a counterexample to the problem statement, since $F_i = G_1 = 5 \mid 95$ and $95 \neq 2^k - 1$ for integer k. This is also in fact the smallest such counterexample. $\square$

Problem 4. Let $\triangle ABC$ be a triangle, and let L be the point on line BC such that AL be a bisector of $\angle BAC$. Let D be the midpoint of AL, and E be the projection of D to AB. Given that $AC = 3AE$, prove that $\triangle CEL$ is an isosceles triangle.



Solution. Let F be the projection of D onto AC, and let P be the projection of L onto AC. By AAS ($\angle AED = \angle AFD = 90^\circ$, $\angle EAD = \angle AFD$, and $AD = AD$),

$$\triangle AED \cong \triangle AFD,$$

so $AE = AF$. Thus, by ASA (since $AE = AF$, $\angle EAL = \angle FAL$, and $AL = AL$),

$$\triangle AEL \cong \triangle AFL,$$

so $LE = LF$. Now we want to show that $LF = LC$. Since DF and PL are both perpendicular to AC, they are parallel to each other, so we also know that

$$\triangle ADF \sim \triangle ALP.$$

Thus,

$$\frac{AF}{AP} = \frac{AD}{AL} = \frac{1}{2},$$

so $AE = AF = FP$. Furthermore, since we are given that $AC = 3AE$, we find that

$$AE = AF = FP = PC.$$

Thus, P is a perpendicular bisector of FC, so $LF = LC$. Thus, we have that $LE = LF = LC$, so $\triangle CEL$ is isosceles, as desired. $\square$

Problem 5. Consider the set S containing all the subsets of $\{1, 2, 3 \cdots n\}$ of size 2. Consider another set T of all natural numbers between 1 to $\frac{n^2-n}{2}$ inclusive. For which n there exist a bijection $f : S \mapsto T$ such that for any non-disjoint sets $a, b \in S$ with $a \neq b$, $(n-1) \nmid f(a) - f(b)$?

Answer. Any even n

Solution. Notice that the condition for the function is equivalent to the following:

Let $g_k : S \setminus \{k\} \to T$ be defined by $g_k(a) = f(\{a, k\})$. Then for all $1 \leq k \leq n$, it must be the case that for all distinct i, j in the domain of g_k,

$$g_k(i) \not\equiv g_k(j) \pmod{n-1}.$$

We can also visualize this by drawing a box where the value inside the cell in the kth row and ath column is $g_k(a)$, and the cells where $k = a$ are crossed out. Then any two cells in a row cannot have the same residue not $n-1$.

Since the size of the domain of g_k is $n-1$ (since it's $S \setminus \{k\}$), it follows that all values of $g_k(a)$ must be distinct mod $n-1$. In other words, the range of g_k $\pmod{n-1}$ is $\{0, 1, \ldots, n-2\}$.

Thus, for each k, every residue mod $n-1$ appears exactly once in the range of g_k, so across all $1 \leq k \leq n$, every residue will appear a total of n times. Notice that if we combine the ranges of all g_k, we will get every element of T twice since every $f(\{k, a\})$ appears exactly twice: once from $g_k(a)$ and once from $g_a(k)$. Thus, every residue mod $n-1$ must appear in T exactly $\frac{n}{2}$ times, and this is not possible when n is odd.

We can also think of this visually by noting that each row has each residue once, and there are n rows, so each residue mod $n-1$ appears n times in the grid. However, since $g_k(a) = g_a(k) = f(\{a, k\})$, the grid is also symmetrical about the line $k = a$, so the sets of numbers above and below the line should be identical. Thus, each residue should appear $\frac{n}{2}$ times in each set, but this is impossible.

Now, we would like to show that such a function does exist for even n. We claim that for $a < b$,

$$f(\{a, b\}) \pmod{n-1} \equiv \begin{cases} a + b \pmod{n-1} & b \neq n \\ 2a \pmod{n-1} & b = n \end{cases}$$

will satisfy the given conditions. We consider the value of $f(\{a, k\})$, where a is held constant. First consider the case where $a \neq n$; if $k < a$, then the outputs range from $a+1, a+2, \ldots, 2a-1 \pmod{n-1}$. If $a < k < n$, the outputs range from $2a+1, \ldots, a+n-1 \equiv a \pmod{n-1}$. If $k = n$, then we get $2a$. Thus, we get $n-1$ consecutive outputs mod $n-1$, so they must all be different.

The case where $a = n$ is similar; we simply get $2k$ as our outputs. Since n is even, $n-1$ will be odd, so $\{0, 2, 4, \ldots, 2(n-2)\} \equiv \{0, 1, 2, \ldots, n-2\} \pmod{n-1}$, so all the outputs will be distinct mod $n-1$ as well.

Lastly, from our argument before, we get that each residue mod $n-1$ will appear

$\frac{n}{2}$ times in our outputs. Since n is even, this is possible, and the numbers from 1 to $\frac{n^2-n}{2}$ indeed has each residue mod $n-1$ appear $\frac{n}{2}$ times. Thus, we can always assign our outputs to elements of T, since only the residues mod $n-1$ are what matter.

The following is a visual representation of the construction for $n = 10$.

10	2	4	6	8	1	3	5	7	0	
9	1	2	3	4	5	6	7	8		0
8	0	1	2	3	4	5	6		8	7
7	8	0	1	2	3	4		6	7	5
6	7	8	0	1	2		4	5	6	3
5	6	7	8	0		2	3	4	5	1
4	5	6	7		0	1	2	3	4	8
3	4	5		7	8	0	1	2	3	6
2	3		5	6	7	8	0	1	2	4
1		3	4	5	6	7	8	0	1	2
	1	2	3	4	5	6	7	8	9	10

$\square$