1. We call a time on a 12 hour digital clock *nice* if the sum of the minutes digits is equal to the hour. For example, 10:55, 3:12 and 5:05 are nice times. How many nice times occur during the course of one day? (We do not consider times of the form 00:XX.)

   Answer: 112

   Solution: Every minute time except 00, 49, 58, 59 has sum between 1 and 12 and so must correspond to exactly one nice time. Therefore, there are $(60-4)\cdot 2 = 112$ nice times (accounting for morning and afternoon).

2. Along Stanford's University Avenue are 2023 palm trees which are either red, green, or blue. Let the positive integers R, G, B be the number of red, green, and blue palm trees respectively. Given that

$$R^3 + 2B + G = 12345,$$

   compute R.

   Answer: 21

   Solution: There are many ways to do this problem. The easiest approach is probably by trying to bound the values. Since R is cubed, it is easiest to bound quickly.

   For the upper bound of R, if $R \geq 24$ then the LHS will exceed the RHS, regardless of the choice of B, G. For the lower bound of R, if $R \leq 20$ then the RHS will exceed the LHS, regardless of the choice of B, G.

   It remains to try the only two options left ($R = 21$, $R = 22$, and $R = 23$). $R = 21$ gives a solution, and $R = 22$, $R = 23$, are easy to reject with some basic algebra.

3. 5 integers are each selected uniformly at random from the range 1 to 5 inclusive and put into a set S. Each integer is selected independently of the others. What is the expected value of the minimum element of S?

   Answer: $\frac{177}{125}$

   Solution 1: The probability that the value of the minimum element is i is $\left(\frac{6-i}{5}\right)^5 - \left(\frac{5-i}{5}\right)^5$ since $\left(\frac{6-i}{5}\right)^5$ is the probability that all the integers selected are at least i and we must subtract the probability $\left(\frac{5-i}{5}\right)^5$ that all the integers selected are at least $i+1$. This gives us

$$\sum_{i=1}^{5} i\left(\left(\frac{6-i}{5}\right)^5 - \left(\frac{5-i}{5}\right)^5\right) = \sum_{i=1}^{5}\left(\frac{i}{5}\right)^5 = \boxed{\frac{177}{125}}.$$

   Solution 2:

$$\mathbb{E}[\min(S)] = \sum_{j=1}^{5}(j\Pr(\min(S) = j))$$

   The challenge lies in finding an expression for the probability. Let the event X_j be the event that no number smaller than j is chosen. Let the event Y_j be the event that the number j is chosen at least once. We will use the notation $\Pr(A, B)$ to denote the probability of some two events A and B both occurring. We will use the notation $\Pr(A|B)$ to denote the probability of an event A occurring given that an event B has already occurred. Then:

$$\begin{aligned}\Pr(\min(S)=j) &= \Pr(X_j, Y_j)\\ &= \Pr(X_j)\Pr(Y_j|X_j)\end{aligned}$$

We can do the rest with combinatorics. There are 5^5 ways to choose the 5 numbers, so this is our sample space. For a given j, $(5-j+1)^5$ of those combinations contain no number smaller than j, so this is our event space. Therefore,

$$\Pr(X_j) = \left(\frac{5-j+1}{5}\right)^5 = \left(\frac{6-j}{5}\right)^5$$

If we know that X_j has occurred, then there were only $(5-j+1)^5$ ways to fill S, so this is our sample space given that X_j is true. Of these, we can count that there are $((5-j+1)-1)^5$ ways to choose numbers to fill S such that they do not contain j given that X_j is true. Therefore, our event space should be the total possibilities to fill S given X_j minus the number of ways to fill S that don't include j given X_j is true. This means our event space is: $(5-j+1)^5-((5-j+1)-1)^5$. Therefore,

$$\Pr(Y_j|X_j) = \frac{(5-j+1)^5-((5-j+1)-1)^5}{(5-j+1)^5} = \frac{(6-j)^5-(5-j)^5}{(6-j)^5}$$

Putting it all together,

$$\begin{aligned}\Pr(\min(S)=j) &= \Pr(X_j, Y_j)\\ &= \Pr(X_j)\Pr(Y_j|X_j)\\ &= \left(\frac{6-j}{5}\right)^5\left(\frac{(6-j)^5-(5-j)^5}{(6-j)^5}\right)\\ &= \frac{(6-j)^5-(5-j)^5}{5^5}\end{aligned}$$

Then,

$$\mathbb{E}[\min(S)] = \sum_{j=1}^{5} j\left(\frac{(6-j)^5-(5-j)^5}{5^5}\right)$$

After simplification, we can re-write this as the beautiful:

$$\sum_{j=1}^{5}\left(\frac{j}{5}\right)^5 = \boxed{\frac{177}{125}}.$$

4. Cornelius chooses three complex numbers a, b, c uniformly at random from the complex unit circle. Given that real parts of $a\cdot\overline{c}$ and $b\cdot\overline{c}$ are $\frac{1}{10}$, compute the expected value of the real part of $a\cdot\overline{b}$.

Answer: $\frac{1}{100}$

Solution: Treating these as vectors, this weird real part is actually the dot product of the vectors. Now, note that this is equivalent to Cornelius first choosing $c=1$ and then randomly choosing a and b. Given that both of them are $\frac{1}{10}$ it follows that either $a=b$ or they are on opposite sides of 1, each having $\cos(\theta)=\frac{1}{10}$, where θ is the angle between a, c (and b, c). Then, our answer is $\frac{1}{2}\cdot 1+\frac{1}{2}\cdot\cos(2\theta) = \frac{1}{2}+\frac{1}{2}\left(2\cos^2\theta-1\right) = \cos^2\theta = \boxed{\frac{1}{100}}$.

5. A computer virus starts off infecting a single device. Every second an infected computer has a 7/30 chance to stay infected and not do anything else, a 7/15 chance to infect a new computer, and a 1/6 chance to infect two new computers. Otherwise (a 2/15 chance), the virus gets exterminated, but other copies of it on other computers are unaffected. Compute the probability that a single infected computer produces an infinite chain of infections.

Answer: $\frac{4}{5}$

Solution: Let P denote the probability that a single infected computer produces an infinite chain of infections. One second after the first computer gets infected, there is a 7/30 chance for nothing to change; in this case the probability to produce an infinite chain of infections is still P. Next, there is a 14/30 chance for there to be two infected computers; there is a $1-(1-P)^2 = 2P - P^2$ (by complementary counting or inclusion-exclusion) chance for there to be an infinite chain of infections from this state. Finally, there is a 5/30 chance for there to be three infected computers; there is a $1-(1-P)^3 = 3P - 3P^2 + P^3$ (by complementary counting or inclusion-exclusion) chance for there to be an infinite chain of infections from this state. Then, we can set up the relation

$$P = \frac{7}{30}P + \frac{14}{30}(2P - P^2) + \frac{5}{30}(3P - 3P^2 + P^3).$$

Dividing through by P, multiplying both sides by 30, and doing some rearrangement gives

$$0 = 5P^2 - 29P + 20.$$

Applying the quadratic formula produces

$$P = \frac{29 \pm \sqrt{29^2 - 4 \cdot 5 \cdot 20}}{2 \cdot 5} = \frac{29 \pm \sqrt{841 - 400}}{10} = \frac{29 \pm 21}{10}.$$

Now, note that if we take the positive branch, P evaluates to 5, which cannot be a probability; thus we take the negative branch and find

$$P = \frac{29 - 21}{10} = \frac{8}{10} = \boxed{\frac{4}{5}}.$$

6. In the language of *Blah*, there is a unique word for every integer between 0 and 98 inclusive. A team of students has an unordered list of these 99 words, but do not know what integer each word corresponds to. However, the team is given access to a machine that, given two, not necessarily distinct, words in *Blah*, outputs the word in *Blah* corresponding to the sum modulo 99 of their corresponding integers. What is the minimum N such that the team can narrow down the possible translations of "1" to a list of N *Blah* words, using the machine as many times as they want?

Answer: 60

Solution: We can only narrow down 1 to the list of *Blah* words which are relatively prime to 99. Any number which is not relatively prime we can distinguish from 1 using the machine, since we can add it to itself repeatedly until we get to 0 (when we hit the original word again, we know the word before that meant 0). Any number not relatively prime to 99 will repeat after fewer than 99 iterations, which distinguishes it from 1.

On the other hand, we cannot distinguish any integer k which is relatively prime to 99 from 1. This is because the map $x \mapsto kx$ on the integers modulo 99 will be a bijection which preserves addition (i.e. $k(a+b) \equiv ka + kb$). Therefore, the "addition table" will be the same even if we replaced each number x with kx.

Thus, the best we can do is narrow it down to $N = \varphi(99)$ possibilities. Therefore, the answer is $\varphi(99) = \varphi(9) \cdot \varphi(11) = 6 \cdot 10 = \boxed{60}$.

7. Compute

$$\sqrt{6\sum_{t=1}^{\infty}\left(1+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(1+k)^{-j}\right)^2\right)^{-t}}.$$

Answer: $\mathbf{\frac{6}{\pi}}$

Solution: Using sum of geometric series:

$$\sum_{j=1}^{\infty}(1+k)^{-j} = \frac{1}{k}$$

Using sum of squared reciprocals:

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(1+k)^{-j}\right)^2 = \sum_{k=1}^{\infty}\left(\frac{1}{k}\right)^2 = \frac{\pi^2}{6}$$

Using sum of geometric series:

$$\sum_{t=1}^{\infty}\left(1+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(1+k)^{-j}\right)^2\right)^{-t} = \sum_{t=1}^{\infty}\left(1+\frac{\pi^2}{6}\right)^{-t} = \frac{6}{\pi^2}$$

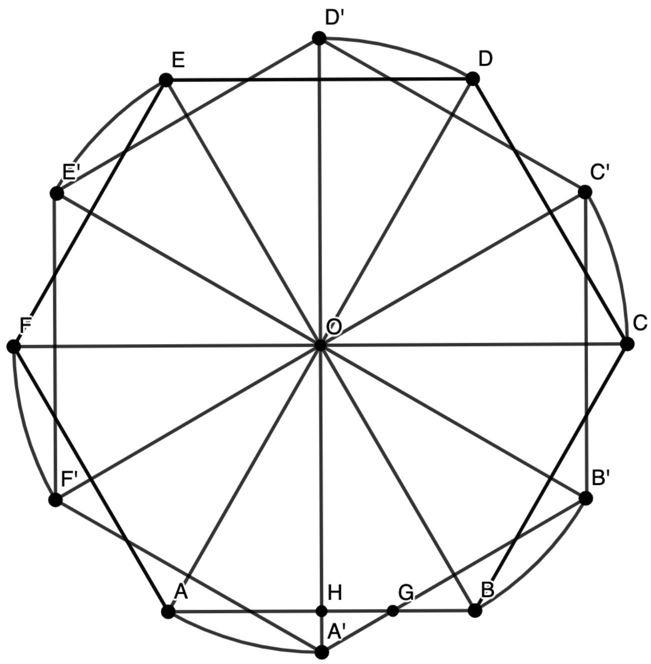
Then the rest is simply

$$\sqrt{6\frac{6}{\pi^2}} = \frac{6}{\pi}$$

8. What is the area that is swept out by a regular hexagon of side length 1 as it rotates 30° about its center?

Answer: $\mathbf{\frac{\pi}{2} + 6\sqrt{3} - 9}$

Solution: The resulting figure that the hexagon covers as it rotates can be divided into 12 sections, 6 of which are circle sectors with radius 1 and an angle of 30°. The total area of these sectors is $\frac{\pi}{2}$.



To visualize the other 6 sections, consider a hexagon with vertices A, B, C, D, E, and F and let $A'B'C'D'E'F'$ let be the image of $ABCDEF$ from a $30°$ counterclockwise rotation about its center, which we denote as point O. Let the intersection of AB and $A'B'$ be G. Then, each of the 6 sections is congruent to quadrilateral $OA'GB$. We can compute the area of $OA'GB$ by splitting it into triangles $OA'G$ and OBG, which are congruent to each other. We have that $OA' = 1$. Let the intersection of AB and OA' be H. Since AOH is a 30-60-90 triangle, $OH = \frac{\sqrt{3}}{2}$. Now, note that triangle HOG is a right triangle with $\angle HOG = 15°$ and $\angle HGO = 75°$, so $OG = \frac{OH}{\sin 75°} = \frac{\sqrt{3}}{2\sin 75°}$. Then, the area of triangle $A'OG$ is

$$\begin{aligned}\frac{1}{2}(OA')(OG)\sin\angle A'OG &= \frac{1}{2}\cdot\frac{\sqrt{3}}{2\sin 75°}\cdot\sin 15° \\ &= \frac{\sqrt{3}}{4}\tan(15°) \\ &= \frac{\sqrt{3}}{4}\sqrt{\frac{1-\sqrt{3}/2}{1+\sqrt{3}/2}} \\ &= \frac{2\sqrt{3}-3}{4}.\end{aligned}$$

Thus, the area of $OA'GB$ is $\frac{2\sqrt{3}-3}{2}$. The total area of the 6 sections is then $6\sqrt{3}-9$. For the area swept out by the hexagon, we have $\boxed{\frac{\pi}{2}+6\sqrt{3}-9}$.

9. Let A be the the area enclosed by the relation

$$x^2 + y^2 \le 2023.$$

Let B be the area enclosed by the relation

$$x^{2n} + y^{2n} \leq \left(A \cdot \frac{7}{16\pi}\right)^{n/2}.$$

Compute the limit of B as $n \to \infty$ for $n \in \mathbb{N}$.

Answer: 119

Solution: A is the area of a circle with radius of $\sqrt{2023}$, so A is 2023π. The shape described in B is a square with side length $2\left(A \cdot \frac{7}{16\pi}\right)^{1/4}$. Therefore, B is equal to $4\sqrt{2023\pi \cdot \frac{7}{16\pi}} = 4\sqrt{\left(\frac{119}{4}\right)^2} = \boxed{119}$.

10. Let $\mathcal{S} = \{1, 6, 10, \dots\}$ be the set of positive integers which are the product of an even number of distinct primes, including 1. Let $\mathcal{T} = \{2, 3, \dots,\}$ be the set of positive integers which are the product of an odd number of distinct primes.

Compute

$$\sum_{n \in \mathcal{S}} \left\lfloor \frac{2023}{n} \right\rfloor - \sum_{n \in \mathcal{T}} \left\lfloor \frac{2023}{n} \right\rfloor.$$

Answer: 1

Solution: We prove by induction that for all $k \geq 1$,

$$\sum_{n \in \mathcal{S}} \left\lfloor \frac{k}{n} \right\rfloor - \sum_{n \in \mathcal{T}} \left\lfloor \frac{k}{n} \right\rfloor = 1$$

The base case $k = 1$ is clear.

Now, suppose the theorem holds for k. For $k+1$, each term $\left\lfloor \frac{k+1}{n} \right\rfloor$ has increased iff n is a factor of $k+1$. The first sum increases by the number of factors of $k+1$ that are in $\mathcal{S}$, while the second sum increases by the number of factors of $k+1$ that are in $\mathcal{T}$. Thus, we simply need to show these are the same.

Since $k+1 \geq 2$, there is at least one prime factor $p \mid k+1$. We can pair the factors of $k+1$ in $\mathcal{S}$ with those in $\mathcal{T}$ by pairing $a \mid k+1$ with pa if $p \nmid a$ or a/p if $p \mid a$ (essentially toggling whether p is a factor of a). This flips the parity of the number of distinct prime factors, and is invertible, so $k+1$ has the same number of factors in $\mathcal{S}$ and $\mathcal{T}$.

This completes the induction, so the answer is $\boxed{1}$.

11. Define the Fibonacci sequence by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Compute

$$\lim_{n \to \infty} \frac{F_{F_{n+1}+1}}{F_{F_n} \cdot F_{F_{n-1}-1}}.$$

Answer: $\mathbf{\frac{5+3\sqrt{5}}{2}}$

Solution: The key step is to recall/rederive that $F_n \sim \frac{1}{\sqrt{5}}\varphi^n$ for $\varphi = \frac{1+\sqrt{5}}{2}$. With these asymptotics, our limit reduces to

$$\lim_{n \to \infty} \frac{F_{F_{n+1}+1}}{F_{F_n} \cdot F_{F_{n-1}-1}} = \lim_{n \to \infty} \frac{\frac{1}{\sqrt{5}}\varphi^{F_{n+1}+1}}{\frac{1}{\sqrt{5}}\varphi^{F_n} \cdot \frac{1}{\sqrt{5}}\varphi^{F_{n-1}-1}} = \sqrt{5}\varphi^2 = \boxed{\frac{5+3\sqrt{5}}{2}}.$$

12. Let A, B, C, and D be points in the plane with integer coordinates such that no three of them are collinear, and where the distances AB, AC, AD, BC, BD, and CD are all integers. Compute the smallest possible length of a side of a convex quadrilateral formed by such points.

Answer: 3

Solution: We want to find the minimum length of any side. WLOG we minimize AB. Then, consider the triangle $\triangle ABC$. WLOG we can take $A = (0,0)$ and $B = (AB, 0)$ (since we can rotate the points to this and preserve the fact they have integer coordinates). Using the triangle inequality, we have $|AC - BC| \leq AB$ with equality if A, B, C are collinear. If $AB = 1$, then $AC = BC$ and $C = (1/2, y)$, which does not have integer coefficients

Now if $AB = 2$ and $C = (x, y)$, $AC^2 = x^2 + y^2$ and $BC^2 = (2-x)^2 + y^2$. Since $|AC - BC|$ is 0 or 1, and AC^2 and BC^2 have the same parity, they must be equal. So, $x = 1$. But then $1 + y^2 = AC^2$ where y, AC are both integers. This only occurs if $y = 0$, but then A, B, C are collinear, a contradiction.

So, the minimal length of AB is $\boxed{3}$ and we can construct this with $A = (0,0)$, $B = (3,0)$, $C = (0,4)$ and $D = (3,4)$.

13. Suppose the real roots of $p(x) = x^9 + 16x^8 + 60x^7 + 1920x^2 + 2048x + 512$ are $r_1, r_2 \ldots, r_k$ (roots may be repeated). Compute

$$\sum_{i=1}^{k} \frac{1}{2 - r_i}.$$

Answer: $\mathbf{\frac{5}{4}}$

Solution: We transform $x \to 2x$ to obtain that the roots are twice those of $f(x) = x^9 + 8x^8 + 15x^7 + 15x^2 + 8x + 1$. Note that the final computation is now

$$\frac{1}{2} \sum_{i=1}^{k} \frac{1}{1 - r_i}$$

where r_i are the roots of this new polynomial.

Note that -1 is a root of this polynomial. Our goal is characterize the other roots. By noting that $f(x)$ is palindromic (as in, $f(x) = x^9 f(1/x)$) it follows that roots must come in conjugate pairs: so, if x is a root, then so is $\frac{1}{x}$.

So, it follows that as for any pair $(x, 1/x)$ we have $\frac{1}{1-x} + \frac{1}{1-\frac{1}{x}} = 1$, so we can think of each root x as contributing a value of $\frac{1}{4}$ to the sum. Therefore, we only care about the *number* of real roots. We show that there are 5 of these, and so the answer is $\frac{5}{4}$.

Now, let's divide by $x + 1$. This yields

$$g(x) = x^8 + 7x^7 + 8x^6 - 8x^5 + 8x^4 - 8x^3 + 8x^2 + 7x + 1$$

which does not have -1 as a root. As $f(x)$ had no negative coefficients, it had no nonnegative real roots and so neither does g. Suppose the negatives of the pairs of roots of g are $s_1, \frac{1}{s_1}, \ldots, s_4, \frac{1}{s_4}$ (so that any negative real roots become positive).

By Descartes Rule of Signs, the number of negative real solutions is at most the number of sign changes in

$$g(-x) = x^8 - 7x^7 + 8x^6 + 8x^5 + 8x^4 + 8x^3 + 8x^2 - 7x + 1$$

which is 4. Hence, the number of real solutions is either 0, 2, or 4.

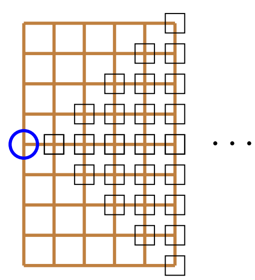
Note however that $g(-1) > 0$ and $g(0) > 0$, and $g(x)$ has no repeated roots (by taking the greatest common factor of g and g'). Hence, there are either no roots or 2 roots in the range $[-1, 0]$.

Our next step is to show there is at least one root in this range. Probably the easiest way to see this is to do a sweep from $x = 0$ to $x = -1$. Note that for $0 \geq x \geq -\frac{2}{7}$, we have

$$x^8 + 8x^2 - 8x^3 + 8x^4 - 8x^5 + 8x^6 = 8x^2(1 + |x| + |x|^2 + |x|^3 + |x|^4) = 8x^2 \cdot \frac{1 - |x|^5}{1 - |x|} \leq \frac{56}{5}x^2 \leq \frac{32}{35}$$

and so $1 + x^8 + \frac{32}{35} < 2$ and hence the contribution of all positive terms is strictly less than 2 in this interval. Certainly, $g(0) = 1 > 0$. However, $g(-\frac{2}{7}) < 0$ by the above logic so there must exist a real solution in $[0, -\frac{2}{7}]$. Therefore, there are a total of 4 real solutions and we have shown our answer of $\frac{5}{4}$.

14. A teacher stands at $(0, 10)$ and has some students, where there is exactly one student at each integer position in the following triangle:



Here, the circle denotes the teacher at $(0, 10)$ and the triangle extends until and includes the column $(21, y)$.

A teacher can see a student (i, j) if there is no student in the direct line of sight between the teacher and the position (i, j). Compute the number of students the teacher can see (assume that each student has no width—that is, each student is a point).

Answer: 279

Solution: We count the number of slopes between $(0, n)$ and (i, j). Since each of these radiate from the same point on one end, this will count the number of students the teacher can see. To do so, modify the grid: put the teacher at $(0, 0)$ and the students at (x, y) with $1 \leq x \leq 21$, $-20 \leq y \leq 20$ in the triangle shape.

Then, the slope of a line is exactly $\frac{y}{x}$. This implies that we are counting the number of reduced fractions $\frac{y}{x}$ where (x, y) are in the triangle. Restrict our attention to the positive triangle: that is, the set of points $1 \leq y < x \leq 21$, and let s be the number of reduced fractions.

Then, the number of reduced fractions in this set is equal to the number of reduced fractions in the set $1 \leq x < y \leq 21$. Hence, we find that $2s + 1$ is the total number of reduced fractions in the set $1 \leq x, y \leq 21$, which is

$$\sum_{i=1}^{21} \sum_{j=1}^{21} \mathbf{1}[\gcd(i, j) = 1].$$

This latter expression is the indicator of the gcd being 1: it is 1 if the gcd is 1 and 0 otherwise.

Furthermore, note that our final answer is $2s+1$ as well (by symmetry and the midline), so we compute this value.

Note that since $\sum_{i=1}^{n} \mathbf{1}[\gcd(i,n)=1] = \varphi(n)$ and $\mathbf{1}[\gcd(n,n)=1] = 0$, this implies that we can write

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{1}[\gcd(i,j)=1] = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} \mathbf{1}[\gcd(i,j)=1] + 2\varphi(n).$$

Recursing then yields that

$$\sum_{i=1}^{21}\sum_{j=1}^{21} \mathbf{1}[\gcd(i,j)=1] = 1 + 2\sum_{i=2}^{21} \varphi(i).$$

Computing this last sum is a little bit tedious, but made easier by observing that for odd x, $\varphi(2x) = \varphi(x)$ whereas for even x, $\varphi(2x) = 2\varphi(x)$.

$$\sum_{i=2}^{21} \varphi(n) = 1+2+2+4+2+6+4+6+4+10+4+12+6+8+8+16+6+18+8+12 = 139.$$

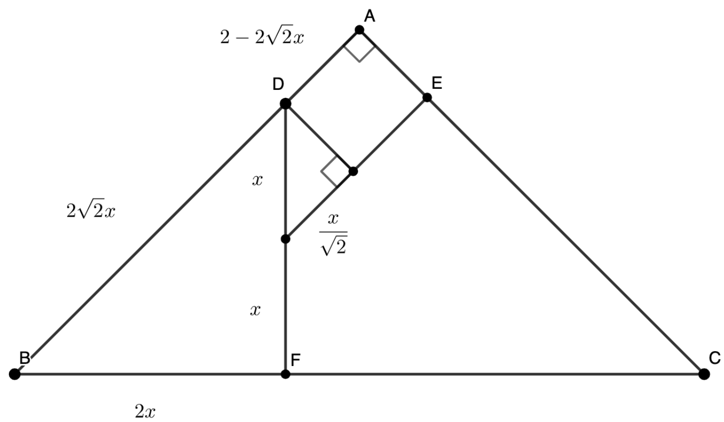
Hence, our answer is $139 \cdot 2 + 1 = 279$.

15. Suppose we have a right triangle $\triangle ABC$ where A is the right angle and lengths $AB = AC = 2$. Suppose we have points D, E, and F on AB, AC, and BC respectively with $DE \perp EF$. What is the minimum possible length of DF?

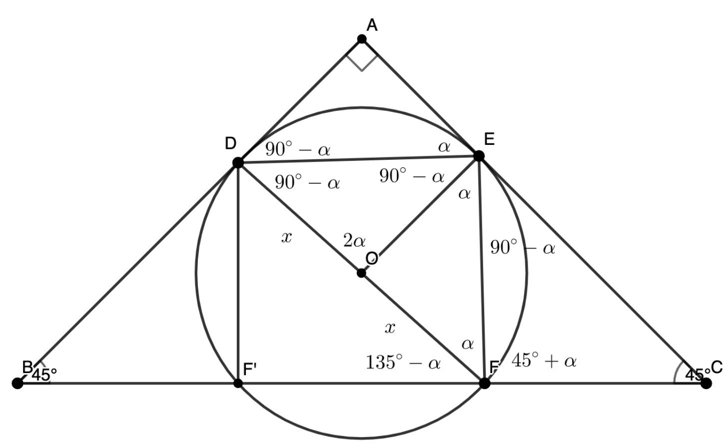
Answer: $\mathbf{\sqrt{5} - 1}$

Solution: Suppose that point D is fixed and let the midpoint of DF be O. Then, the minimum length of DF is achieved by choosing point F on BC such that EO is perpendicular to AC. To see this, note that minimizing DF is equivalent to minimizing the distance between F and the foot of the altitude from D to BC. We consider two cases. Denote the circle with diameter DF as $\odot O$ and note that E must be one of the intersections of $\odot O$ with AC in order for $\angle DEF = 90^\circ$.

Case 1: If we let F be the foot of the altitude from D to BC, $\odot O$ intersects AC. We get the minimum length of DF for which this occurs when $\odot O$ is tangent to AC, as otherwise we can move D farther from A along side AB to reduce the length of DF. In this case, let $DF = 2x$. Then, $BD = 2\sqrt{2}x$ and $AD = 2 - 2\sqrt{2}x$. Let the foot of the perpendicular from D to OE be G. Then, $OE = OG + GE = x/\sqrt{2} + 2 - 2\sqrt{2}x$. Then, since E lies on $\odot O$, we have $x/\sqrt{2} + 2 - 2\sqrt{2}x = x$. Solving for x gives us $\frac{6\sqrt{2}-4}{7}$, so $DF = \frac{12\sqrt{2}-8}{7}$ in this case.



Case 2: Denote the foot of the altitude from D to BC as F' and again let $DF = 2x$. In this case we consider if the circle with diameter DF' does not intersect AC. Again, the minimum value of the length of DF is achieved when $\odot O$ is tangent to AC, since otherwise, we can move F closer to F' and reduce the length of DF while still having $\odot O$ intersect AC at some point. Let $\angle DFE = \alpha$. Then, $\angle OEF = \alpha$, so $\angle EOF = 180^\circ - 2\alpha$. By Law of Sines on $\triangle EOF$, we have $EF = \sin(2\alpha) \cdot \frac{x}{\sin\alpha} = 2x\cos\alpha$. Next, note that $\angle CEF = 90^\circ - \alpha$ and $\angle ECF = 45^\circ$. By Law of Sines on $\triangle CEF$, we have $CF = \cos\alpha \cdot \frac{2x\cos\alpha}{\sin 45^\circ} = 2\sqrt{2}x\cos^2\alpha$. We also know $\angle DBF = 45^\circ$ and can angle chase to find that $BDF = 2\alpha$. By Law of Sines on $\triangle BDF$, we have $BF = \sin(2\alpha) \cdot \frac{2x}{\sin 45^\circ} = 2\sqrt{2}x\sin(2\alpha)$. Now we see that $BF + CF = BC$, which gives us $2\sqrt{2}x\sin(2\alpha) + 2\sqrt{2}x\cos^2\alpha = 2\sqrt{2}$.



We get $x = \frac{1}{\sin(2\alpha)+\cos^2\alpha}$, so now we seek to maximize $\sin(2\alpha)+\cos^2\alpha$ in order to minimize x. We have $\sin(2\alpha) + \cos^2\alpha = \sin(2\alpha) + \frac{\cos(2\alpha)+1}{2}$. Let $\beta \in [0, \pi]$ such that $\sin\beta = \frac{2}{\sqrt{5}}$ and $\cos\beta = \frac{1}{\sqrt{5}}$. Then, $\sin(2\alpha) + \frac{\cos(2\alpha)}{2} + \frac{1}{2} = \frac{\sqrt{5}}{2}(\sin(2\alpha)\sin\beta + \cos(2\alpha)\cos\beta) + \frac{1}{2} = \frac{\sqrt{5}}{2}\cos(2\alpha - \beta) + \frac{1}{2}$, so the maximum possible value is $\frac{\sqrt{5}+1}{2}$, achieved when $2\alpha = \beta$. The minimum value of x is then $\frac{1}{\frac{\sqrt{5}+1}{2}} = \frac{1}{2}(\sqrt{5}-1)$, which gives us the minimum length of DF as $\sqrt{5}-1$. This is smaller than the length found in case 1, so our answer is $\boxed{\sqrt{5}-1}$.