Time limit: 50 minutes.
Instructions: For this test, you work in teams to solve 15 short answer questions. All answers must be expressed in simplest form unless specified otherwise.
No calculators.

1. We call a time on a 12 hour digital clock *nice* if the sum of the minutes digits is equal to the hour. For example, 10:55, 3:12 and 5:05 are nice times. How many nice times occur during the course of one day? (We do not consider times of the form 00:XX.)

2. Along Stanford's University Avenue are 2023 palm trees which are either red, green, or blue. Let the positive integers R, G, B be the number of red, green, and blue palm trees respectively. Given that
$$R^3 + 2B + G = 12345,$$
compute R.

3. 5 integers are each selected uniformly at random from the range 1 to 5 inclusive and put into a set S. Each integer is selected independently of the others. What is the expected value of the minimum element of S?

4. Cornelius chooses three complex numbers a, b, c uniformly at random from the complex unit circle. Given that real parts of $a \cdot \overline{c}$ and $b \cdot \overline{c}$ are $\frac{1}{10}$, compute the expected value of the real part of $a \cdot \overline{b}$.

5. A computer virus starts off infecting a single device. Every second an infected computer has a 7/30 chance to stay infected and not do anything else, a 7/15 chance to infect a new computer, and a 1/6 chance to infect two new computers. Otherwise (a 2/15 chance), the virus gets exterminated, but other copies of it on other computers are unaffected. Compute the probability that a single infected computer produces an infinite chain of infections.

6. In the language of *Blah*, there is a unique word for every integer between 0 and 98 inclusive. A team of students has an unordered list of these 99 words, but do not know what integer each word corresponds to. However, the team is given access to a machine that, given two, not necessarily distinct, words in *Blah*, outputs the word in *Blah* corresponding to the sum modulo 99 of their corresponding integers. What is the minimum N such that the team can narrow down the possible translations of "1" to a list of N *Blah* words, using the machine as many times as they want?

7. Compute
$$\sqrt{6\sum_{t=1}^{\infty}\left(1+\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}(1+k)^{-j}\right)^2\right)^{-t}}.$$

8. What is the area that is swept out by a regular hexagon of side length 1 as it rotates 30° about its center?

9. Let A be the the area enclosed by the relation
$$x^2 + y^2 \leq 2023.$$

Let B be the area enclosed by the relation

$$x^{2n} + y^{2n} \leq \left(A \cdot \frac{7}{16\pi}\right)^{n/2}.$$

Compute the limit of B as $n \to \infty$ for $n \in \mathbb{N}$.

10. Let $\mathcal{S} = \{1, 6, 10, \dots\}$ be the set of positive integers which are the product of an even number of distinct primes, including 1. Let $\mathcal{T} = \{2, 3, \dots,\}$ be the set of positive integers which are the product of an odd number of distinct primes.

    Compute

$$\sum_{n \in \mathcal{S}} \left\lfloor \frac{2023}{n} \right\rfloor - \sum_{n \in \mathcal{T}} \left\lfloor \frac{2023}{n} \right\rfloor.$$

11. Define the Fibonacci sequence by $F_0 = 0$, $F_1 = 1$, and $F_i = F_{i-1} + F_{i-2}$ for $i \geq 2$. Compute

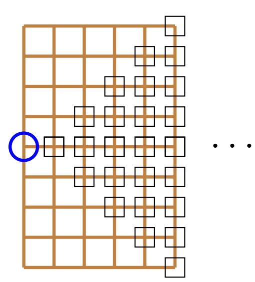
$$\lim_{n \to \infty} \frac{F_{F_{n+1}+1}}{F_{F_n} \cdot F_{F_{n-1}-1}}.$$

12. Let A, B, C, and D be points in the plane with integer coordinates such that no three of them are collinear, and where the distances AB, AC, AD, BC, BD, and CD are all integers. Compute the smallest possible length of a side of a convex quadrilateral formed by such points.

13. Suppose the real roots of $p(x) = x^9 + 16x^8 + 60x^7 + 1920x^2 + 2048x + 512$ are $r_1, r_2 \dots, r_k$ (roots may be repeated). Compute

$$\sum_{i=1}^{k} \frac{1}{2 - r_i}.$$

14. A teacher stands at $(0, 10)$ and has some students, where there is exactly one student at each integer position in the following triangle:



    Here, the circle denotes the teacher at $(0, 10)$ and the triangle extends until and includes the column $(21, y)$.

    A teacher can see a student (i, j) if there is no student in the direct line of sight between the teacher and the position (i, j). Compute the number of students the teacher can see (assume that each student has no width—that is, each student is a point).

15. Suppose we have a right triangle $\triangle ABC$ where A is the right angle and lengths $AB = AC = 2$. Suppose we have points D, E, and F on AB, AC, and BC respectively with $DE \perp EF$. What is the minimum possible length of DF?