# NYCMT 2024-2025 Homework #6

NYCMT

May 9 - May 23, 2025

These problems are due May 23. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve. Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on May 23, you can scan your solutions and email them to ashleyzhu111@gmail.com, sjschool26@gmail.com, and stevenytlou@gmail.com. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or use one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

**Problem 1.** Show that for all positive integers $n$,

$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor = \left\lfloor \sqrt{4n+3} \right\rfloor.$$

**Problem 2.** A finite number of points are marked on the plane, no three of them collinear. A circle is circumscribed around each triangle with marked vertices. Is it possible that centers of all of these circles are also marked?

**Problem 3.** The recursive integer sequence is defined by $F_n = \frac{F_{n-1}-1}{2}$, with $F_1 = n$ (notice this sequence terminates when there are no longer integer values). If there exists an integer $i > 1$ such that $F_i \mid n$, must it be true that $n$ is of the form $2^k - 1$ for integer $k$?

**Problem 4.** Let $\triangle ABC$ be a triangle, and $AL$ be a bisector of $\angle BAC$. Let $D$ be the midpoint of $AL$, and $E$ be the projection of $D$ to $AB$. Given that $AC = 3AE$, prove that $\triangle CEL$ is an isosceles triangle.

**Problem 5.** Consider the set $S$ containing all the subsets of $\{1, 2, 3 \cdots n\}$ of size 2. Consider another set $T$ of all natural numbers between 1 to $\frac{n^2-n}{2}$ inclusive. For which $n$ there exist a bijection $f : S \mapsto T$ such that for any non-disjoint sets $a, b \in S$ with $a \neq b$, $(n-1) \nmid f(a) - f(b)$?