NYCMT 2024-2025 Homework #5 Solutions

NYCMT

February 2025

1 The Usual

Problem 1. Find the value of the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{2^{2^n}+1} + \frac{1}{2^{2^n}-1}.$$

Answer. $\boxed{2/3}$

Solution. Using partial fraction decomposition, we can rewrite

$$\frac{1}{2^{2^n}-1} = \frac{1/2}{2^{2^{n-1}}-1} - \frac{1/2}{2^{2^{n-1}}+1}.$$

Then we can rewrite the whole sum as

$$\begin{aligned}
S &= \sum_{n=1}^{\infty} \frac{1}{2^{2^n}+1} + \frac{1/2}{2^{2^{n-1}}-1} - \frac{1/2}{2^{2^{n-1}}+1} \\
&= -\frac{1}{2^{2^0}+1} + \sum_{n=0}^{\infty} \frac{1}{2^{2^n}+1} + \sum_{n=0}^{\infty} \frac{1/2}{2^{2^n}-1} - \sum_{n=0}^{\infty} \frac{1/2}{2^{2^n}+1} \\
&= -\frac{1}{3} + \sum_{n=0}^{\infty} \frac{1/2}{2^{2^n}+1} + \sum_{n=0}^{\infty} \frac{1/2}{2^{2^n}-1} \\
&= -\frac{1}{3} + \frac{1/2}{2^{2^0}+1} + \frac{1/2}{2^{2^0}-1} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{2^{2^n}+1} + \frac{1}{2^{2^n}-1} \\
&= -\frac{1}{3} + \frac{1}{6} + \frac{1}{2} + \frac{1}{2}S.
\end{aligned}$$

Thus, we get that $S = \frac{1}{3} + \frac{S}{2}$, so $S = \boxed{2/3}$, as desired.

Note: This solution relies on the fact that the sum converges. We can prove this fact rigorously through a comparison test with a geometric series. $\square$

Problem 2. For how many pairs of consecutive integers in the set

$$\{1000, 1001, 1002, \cdots, 2000\}$$

is no carrying over required when they are added?

Answer. $\boxed{156}$

Solution. We want to find the number of pairs $(a-1,\ a)$ such that no carrying over is required when they are added, where $a \in \{1001, 1002, \ldots, 2000\}$. We claim that each digit of a must be in the set $\{0, 1, 2, 3, 4, 5\}$.

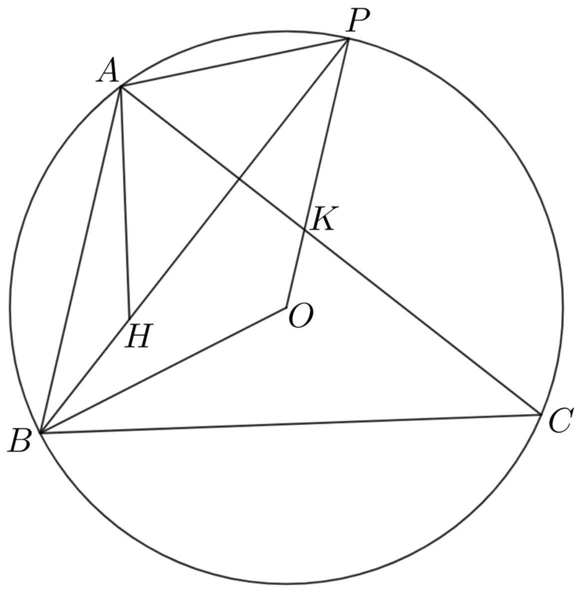
Consider a digit d contained in a. Then either d or $d-1 \pmod{10}$ must be contained within $a-1$, because digits do not skip. Thus, we can manually verify that if $d \in \{0, 1, 2, 3, 4, 5\}$, $d + (d-1 \pmod{10}) < 10$, and if d is a digit outside of this set, this is not true.

Now notice that if digits 0, 1, 2, 3, 4 are in a, it will not cause regrouping when a is added to $a-1$. However, if one of the digits is 5, unless $d-1$ has the digit 4 in the corresponding place value, carrying over *will* be required when they are added. Thus, every digit to the right of the 5 must be 0; otherwise the digit in that place value will not change when 1 is subtracted from a.

If the number does not contain a 5, there are $5^3 = 125$ possibilities; the thousands digit is fixed, and the other 3 digits could be any element of $\{0, 1, 2, 3, 4\}$.
If the number contains a 5 in the ones place, there are $5^2 = 25$ possibilities, since the first and last digits are now fixed.
If the number contains a 5 in the tens place, there are 5 possibilities, since only the hundreds digit is not fixed.
If the number contains a 5 in the hundreds digit, the entire number is fixed, so there is only 1 possibility.

Thus, there are $125 + 25 + 5 + 1 = \boxed{156}$ pairs of consecutive integers in the set satisfying the given property. $\square$

Problem 3. Let $\triangle ABC$ have circumcenter O and orthocenter H, and let it be such that $\angle ABH = \angle HBO$. Let K be the intersection of AC and the line through O parallel to AB. Show that $AH = AK$.



Solution. Let P be the reflection of H over AC (so that $AH = AP$, and $AC \perp BP$). Notice that

$$\begin{aligned}\angle APC &= \angle AHC \\ &= 180^\circ - (\angle CAH + \angle ACH) \\ &= 180^\circ - (90^\circ - \angle C + 90^\circ - \angle A) \\ &= 180 - \angle B.\end{aligned}$$

Thus, P lies on (ABC). We now claim that K lies on line OP. In other words, we want to show that $OP \parallel AB$. Since

$$\angle ABH = \angle HBO = \angle HPO,$$

this is true. Now, notice that

$$\begin{aligned}\angle APK = \angle APO &= \angle PAO \\ &= \frac{1}{2}(180^\circ - \angle AOP) \\ &= \frac{1}{2}(180^\circ - 2\angle ABP) \\ &= \frac{1}{2}(180^\circ - 2(90^\circ - \angle A)) \\ &= \angle A.\end{aligned}$$

Similarly, since $AB \parallel OP$, we can show that

$$\angle AKP = \angle KAB = \angle A.$$

Thus, $\angle APK = \angle AKP$, so $\triangle KAP$ is isosceles with $AK = AP$. Since we also know that $AH = AP$, this implies the desired result.

$\square$

Problem 4. Prove that for all integers $n \geq 3$, there exist odd positive integers x, y such that $7x^2 + y^2 = 2^n$.

Solution. We prove this through mathematical induction on n.

Base Case: $n = 3$
Clearly, $x = y = 1$ is a valid solution to this equation, so the base case holds.

Inductive Hypothesis: We assume that the statement holds for $n = k$ for some integer $k \geq 3$. In other words, we assume that there exist odd positive integers x_k, y_k such that $7x_k^2 + y_k^2 = 2^k$.

Inductive Step: We want to show that the statement holds for $n = k + 1$, so given the inductive hypothesis, we want to show that there exist odd positive integers x_{k+1} and y_{k+1} such that $7x_{k+1}^2 + y_{k+1}^2 = 2^{k+1}$.

By the inductive hypothesis, there exist odd positive integers x_k, y_k such that $7x_k^2 + y_k^2 = 2^k$.. We note that we can rewrite $2^k = 7x_k^2 + y_k^2 = (x_k\sqrt{-7} + y_k)(x_k\sqrt{-7} - y_k)$. We want to find a way to maintain this conjugate pair-esque form while multiplying the total quantity by 2. Because

$$\frac{1 + \sqrt{-7}}{2} \cdot \frac{1 - \sqrt{-7}}{2} = 2$$

and the two factors are conjugates, we can multiply this to our existing expression to get that

$$(x_k\sqrt{-7} + y_k)(x_k\sqrt{-7} - y_k)\left(\frac{1 + \sqrt{-7}}{2}\right)\left(\frac{1 - \sqrt{-7}}{2}\right) = 2^{k+1}.$$

Next, there are two possible ways to pair up the original terms and the terms of the conjugate pair that multiplies to 2. If we pair up $x_k\sqrt{-7} + y_k$ and $\frac{1+\sqrt{-7}}{2}$ and pair the other two factors together, we obtain the equation

$$\left(\frac{x_k + y_k}{2}\sqrt{-7} + \frac{y_k - 7x_k}{2}\right)\left(\frac{x_k + y_k}{2}\sqrt{-7} - \frac{y_k - 7x_k}{2}\right) = 2^{k+1},$$

or we can pair up $x_k\sqrt{-7} + y_k$ and $\frac{1-\sqrt{-7}}{2}$ and pair the other two factors together to obtain the equation

$$\left(\frac{x_k - y_k}{2}\sqrt{-7} + \frac{y_k + 7x_k}{2}\right)\left(\frac{x_k - y_k}{2}\sqrt{-7} - \frac{y_k + 7x_k}{2}\right) = 2^{k+1}.$$

In the first case, we find that

$$(x_{k+1}, y_{k+1}) = \left(\frac{x_k + y_k}{2}, \frac{y_k - 7x_k}{2}\right)$$

is a possible solution to the equation $7x_{k+1}^2 + y_{k+1}^2 = 2^{k+1}$. Meanwhile, the second case tells us that

$$(x_{k+1}, y_{k+1}) = \left(\frac{x_k - y_k}{2}, \frac{y_k + 7x_k}{2}\right)$$

is another possible solution. It remains to show that at least one of those is an odd positive integer solution.

By the inductive hypothesis, both x_k and y_k are odd, so it is clear to see that both solutions produce integers. To confirm that at least one solution produces a pair of odd integers, notice that $\frac{x_k+y_k}{2} + \frac{x_k-y_k}{2} = x_k$. Since those two quantities are integers that add up to an odd integer x_k, one of them must be odd. Now note that

$$\frac{x_k + y_k}{2} - \frac{y_k - 7x_k}{2} = 4x_k = \frac{x_k - y_k}{2} + \frac{y_k + 7x_k}{2}.$$

Thus, in both ordered pairs, x_{k+1} and y_{k+1} must have the same parity (since they add up to an even number). Combining these facts, we can see that one of these possible solutions will return two odd integers. Since our expression involves squares, we can just take their absolute values so that they are both positive.

This completes the proof by induction on n. $\square$

Problem 5. Let $r_1, r_2, \ldots, r_{20}$ be the roots of the polynomial $x^{20} - 7x^3 + 1$. If

$$\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{20}^2+1}$$

can be written in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m+n$.

Answer. $\boxed{240}$

Solution. We will transform the polynomial $P(x) = x^{20} - 7x^3 + 1$ to have roots more similar to the forms in the desired summation. Do note that

$$P(x) = \prod_{i=1}^{20} (x - r_i).$$

First, we would like a new polynomial $Q(x)$ to have roots $r_1^2, r_2^2, \ldots, r_{20}^2$, so

$$\begin{aligned} Q(x) &= \prod_{i=1}^{20} \left(x - r_i^2\right) \\ &= \prod_{i=1}^{20} \left(\sqrt{x} - r_i\right) \cdot \prod_{i=1}^{20} \left(\sqrt{x} + r_i\right) \\ &= P\left(\sqrt{x}\right) \cdot (-1)^{20} P\left(-\sqrt{x}\right) \\ &= \left(x^{10} - 7x^{\frac{3}{2}} + 1\right) \cdot \left(x^{10} + 7x^{\frac{3}{2}} + 1\right) \\ &= x^{20} + 2x^{10} - 49x^3 + 1. \end{aligned}$$

Then, the polynomial $R(x)$ having roots $r_1^2+1, r_2^2+1, \ldots, r_{20}^2+1$ is simply $Q(x-1)$. We now want to find the sum of the reciprocals of the roots of $R(x)$. This can be done in a number of ways, but most directly, if $R(x)$ is written in standard form as $a_{20}x^{20} + \cdots + a_2x^2 + a_1x + a_0$, the answer is $-\frac{a_1}{a_0}$. (This can be shown by reversing the polynomial or as a general consequence of expanding the desired expression.)

We can find

$$a_0 = R(0) = Q(-1) = (-1)^{20} + 2(-1)^{10} - 49(-1)^3 + 1 = 1 + 2 + 49 + 1 = 53.$$

To find the value of a_1, we note that each term of the form $a_n(x-1)^n$ contributes an x term of $a_n \cdot x \cdot (-1)^{n-1} \cdot \binom{n}{1}$. Since

$$R(x) = Q(x-1) = (x-1)^{20} + 2(x-1)^{10} - 49(x-1)^3 + 1,$$

we get a total coefficient of

$$(-1)^{19} \cdot 20 + 2 \cdot (-1)^9 \cdot 10 - 49 \cdot (-1)^2 \cdot 3 = -20 - 20 - 147 = -187.$$

The answer is then $-\frac{-187}{53} = \frac{187}{53}$, so $m + n = 187 + 53 = \boxed{240}$. □

2 Solo Relay!

We start with Problem 8, since its answer gives us the most information.

Problem 6. Let A be the answer to Problem 9. If the value of

$$\sum_{x=A}^{\infty} \frac{1}{x^2 - Ax + (3A-3)}$$

can be expressed as $\frac{1}{n}$, where n is a positive integer, find n.

Solution. From our work on Problem 9, we know that A is either 4 or 13. Thus, the sum we have to evaluate is either

$$\sum_{x=4}^{\infty} \frac{1}{x^2-4x+9} \text{ or } \sum_{x=11}^{\infty} \frac{1}{x^2-11x+30}.$$

We can notice that

$$\frac{1}{x^2-11x+30} = \frac{1}{x-6} - \frac{1}{x-5},$$

so

$$\begin{aligned}\sum_{x=11}^{\infty} \frac{1}{x^2-11x+30} &= \sum_{x=11}^{\infty} \frac{1}{x-6} - \frac{1}{x-5} \\ &= \sum_{x=6}^{\infty} \frac{1}{x-1} - \frac{1}{x} \\ &= \frac{1}{5} + \sum_{x=6}^{\infty} \frac{1}{x} - \sum_{x=6}^{\infty} \frac{1}{x} \\ &= \frac{1}{5}.\end{aligned}$$

Because we cannot perform partial fraction decomposition on $\frac{1}{x^2-4x+9}$ to produce a telescoping sum, it seems unlikely that $A = 4$ is the correct answer. We can confirm this by showing that $\sum_{x=4}^{\infty} \frac{1}{x^2-4x+9} > \frac{1}{3}$, so if $A = 4$ was the correct answer, then n must be 1 or 2. After that, we can confirm that these values of n will not yield an answer less than 10 for Problem 7.

Thus, we find that the answer to Problem 9 was 11, so the answer to Problem 8 was 4. Since the summation evaluates to $\frac{1}{5}$, we find that $\boxed{n = 5}$. ☐

Problem 7. Let B be the answer to Problem 6. Valentines' Day chocolates come in mutually indistinguishable packs of B chocolates and mutually indistinguishable packs of $B+3$ chocolates. How many ways are there to buy exactly 100 chocolates?

Solution. We have that $B = 5$, so we want to find the number of nonnegative integer solutions (x, y) to $5x + 8y = 100$. Note that it must be true that

$$100 - 8y \equiv 0 \pmod 5,$$

which means that y must be divisible by 5 as well. Since x and y are integers, we also have that $8y \leq 100$, so the only possible values of y are 0, 5, and 10. Indeed, we can confirm that there are $\boxed{3}$ solutions: (20, 0); (12, 5); and (4, 10). ☐

Problem 8. Let C be the answer to Problem 7. (It is given that $C < 10$.) Find the tens digit of the 1000th smallest positive integer that doesn't contain C as a digit.

Solution. Since we are looking for the 1000th smallest positive integer that doesn't contain C as a digit, we are essentially working with a modified version of base 9 (since we only have access to 9 digits). However, the set of 9 digits may vary based on the value of C, that is, the digit that is eliminated from $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. We can observe that

$$1000 = 729 + 3 \cdot 81 + 3 \cdot 27 + 1 = 9^3 + 3 \times 9^2 + 3 \times 9 + 1 = 1331_9$$

in standard base 9. However, the digits may change depending on the value of C. In base 9, the digit removed is 9, which causes the tens digit to be 3. Similarly, if $4 \leq C \leq 9$, then the tens digit will still be 3. However, if $1 \leq C \leq 3$, the removed digit will cause the number in the tens digit to shift up and be 4 instead. Lastly, if $C = 0$, the base 9 representation is shifted up one to 1332_9, since the nonpositive integer 0 is not omitted in this case. This still gives us a tens digit of 3. These are the only three possible cases, so we have narrowed down the fact that the answer is either $\boxed{3 \text{ or } 4}$.

Since we got that the answer to Problem 7 was 3, the answer to this question is $\boxed{4}$. This is consistent with the information we have previously obtained. □

Problem 9. Let D be the answer to Problem 8. Ashley rolls a $2D$-sided fair die, as well as two D-sided fair dice. If the probability that the sum of the numbers rolled by the two D-side dice is less than the value rolled by the $2D$-sided die can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers, find $p + q$. (Assume the n sides of an n-sided fair die are numbered from 1 to n.)

Solution. From our analysis on Problem 8, we know that either $D = 3$ or $D = 4$, so we can look at each of those cases.

If $D = 3$, then Ashley is rolling a 6-sided fair die and comparing the value to the sum of the values rolled by two fair 3-sided dice. We can draw the following table for the results when two 3-sided dice are rolled:

Die 2 \ Die 1	1	2	3
1	2	3	4
2	3	4	5
3	4	5	6

Thus, for each possible sum of the two dice rolls, we multiply by the probability we roll a larger value on the 6-sided die. Thus, the desired probability is

$$\frac{1}{9} \cdot \frac{4}{6} + \frac{2}{9} \cdot \frac{3}{6} + \frac{3}{9} \cdot \frac{2}{6} + \frac{2}{9} \cdot \frac{1}{6} + \frac{1}{9} \cdot \frac{0}{6} = \frac{1}{3},$$

since there is a $\frac{1}{9}$ probability the two rolls sum to 2 and a $\frac{4}{6}$ probability of rolling a number greater than 2 on a 6-sided die, etc. In this case, $p + q$=4.

If $D = 4$, then Ashley is rolling an 8-sided die, and comparing the value rolled to the sum of the values rolled by two fair 4-sided dice, which has the following table of results:

Die 2 \ Die 1	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

We can again evaluate the desired probability to be

$$\frac{1}{16} \cdot \frac{6}{8} + \frac{2}{16} \cdot \frac{5}{8} + \frac{3}{16} \cdot \frac{4}{8} + \frac{4}{16} \cdot \frac{3}{8} + \frac{3}{16} \cdot \frac{2}{8} + \frac{2}{16} \cdot \frac{1}{8} + \frac{1}{16} \cdot \frac{0}{8} = \frac{3}{8},$$

and in this case $p + q = 11$.
Thus, the answer is either $\boxed{4 \text{ or } 11.}$ Later, we will be able to deduce that the answer is indeed $\boxed{11}$. □

Solution. For an alternate solution using symmetry, let A be the random variable representing the sum obtained from the two D-sided dice, and B be the random variable representing the sum obtained from the 2D-sided die. Considering the bijection from A to $A - 1$,

$$P(A > B) = P(A - 1 \geq B).$$

However, considering another bijection from A to $2D + 1 - A$,

$$P(A > B) = P(2D + 1 - A < B) = P(A - 1 < B),$$

as the probability distribution of A is symmetric around $D + \frac{1}{2}$. Therefore,

$$2P(A > B) = P(A - 1 \geq B) + P(A - 1 < B) = 1,$$

so $P(A > B) = \frac{1}{2}$. To find $P(A < B)$, we simply need to find $P(A = B)$. This is easy, however, as the two D-sided dice must roll a value between 2 and 2D, and the 2D-sided die has a $\frac{1}{2D}$ chance to match the value. Therefore,

$$P(A < B) = 1 - P(A > B) - P(A = B) = 1 - \frac{1}{2} - \frac{1}{2D} = \frac{D - 1}{2D}.$$

For $D = 3$ or $D = 4$ as given by Problem 8, we get the probability $\frac{1}{3}$ or $\frac{3}{8}$. □

The answers to this section are tabulated below.

Problem Number	**6**	**7**	**8**	**9**
Answer	5	3	4	11