NYCMT 2024-2025 Homework #5

NYCMT

January 31 - February 28, 2025

These problems are due February 28. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve. Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on February 28, you can scan your solutions and email them to ashleyzhu111@gmail.com, sjschool26@gmail.com, and stevenytlou@gmail.com. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or use one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

1 The Usual

Problem 1. Find the value of the following sum:

$$\sum_{n=1}^{\infty} \frac{1}{2^{2^n}+1} + \frac{1}{2^{2^n}-1}.$$

Problem 2. For how many pairs of consecutive integers in the set

$$\{1000, 1001, 1002, \cdots, 2000\}$$

is no carrying over required when they are added?

Problem 3. Let $\triangle ABC$ have circumcenter O and orthocenter H, and let it be such that $\angle ABH = \angle HBO$. Let K be the intersection of AC and the line through O parallel to AB. Show that $AH = AK$.

Problem 4. Prove that for all integers $n \geq 3$, there exist odd positive integers x, y such that $7x^2 + y^2 = 2^n$.

Problem 5. Let $r_1, r_2, \ldots, r_{20}$ be the roots of the polynomial $x^{20} - 7x^3 + 1$. If

$$\frac{1}{r_1^2+1} + \frac{1}{r_2^2+1} + \cdots + \frac{1}{r_{20}^2+1}$$

can be written in the form $\frac{m}{n}$ where m and n are relatively prime positive integers, find $m + n$.

2 Solo Relay!

Problem 6. Let A be the answer to Problem 9. If the value of

$$\sum_{x=A}^{\infty} \frac{1}{x^2 - Ax + (3A - 3)}$$

can be expressed as $\frac{1}{n}$, find n.

Problem 7. Let B be the answer to Problem 6. Valentines' Day chocolates come in mutually indistinguishable packs of B chocolates and mutually indistinguishable packs of $B+3$ chocolates. How many ways are there to buy exactly 100 chocolates?

Problem 8. Let C be the answer to Problem 7. (It is given that $C < 10$.) Find the tens digit of the 1000th smallest positive integer that doesn't contain C as a digit.

Problem 9. Let D be the answer to Problem 8. Ashley rolls a $2D$-sided fair die, as well as two D-sided fair dice. If the probability that the sum of the numbers rolled by the two D-side dice is less than the value rolled by the $2D$-sided die can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers, find $p + q$. (Assume the n sides of an n-sided fair die are numbered from 1 to n.)