NYCMT 2024-2025 Homework #2 Solutions

NYCMT

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Problem 1. There are 2024 math teamers sitting around at a round table. Combo mains always lie, Geo mains always tell the truth, and Alg mains randomly tell the truth or lie. Given that all math teamers are either Combo, Geo, or Alg mains, there is at least one of each type at the table, and you are only allowed to ask yes/no questions to a math teamer regarding the identity of another, is it possible to always identify all 2024 math teamers?

Answer. No

Solution. First, we should note the following observations:
Suppose we ask a Geo main G the three following questions regarding the nature of another math teamer M:

- Is M an Combo main?
- Is M a Geo main?
- Is M an Alg main?

The answer to exactly one of these questions is "yes". Since Geo mains always tell the truth, G will answer "yes" to exactly one of these questions, and "no" to the other two. If a math teamer answers "yes" to "Is M an X main?" (where $X \in \{\text{Combo}, \text{Geo}, \text{Alg}\}$), then we say that they *indicate through truth that M is an X main.* Note that this is only possible when the math teamer is not a Combo main.

Now suppose we ask a Combo main C the same three questions. Since Combo mains always lie, C will answer "no" to exactly one of these questions and "yes" to the other two. So if a math teamer answers "no" to "Is M an X main?", then we say that they *indicate through lies that M is an X* main. Again, note that this is only possible when the math teamer is not a Geo main.

Consider the case where there is exactly one Geo main and one Combo main, and everyone else is an Alg main. We separate the Alg mains into two categories (with size ≥ 2) – "fake Geo mains" and "fake Combo mains".

Suppose that we asked both the Geo G main and a fake Geo main A_G about the other's identity. Then G indicates through truth that A_G is an Alg main, and suppose that A_G indicates through truth that G is an Alg main, and that this occured with all fake Geo mains.

In general, when a real Geo or Combo main is questioned about a fake Geo or Combo main's identity, they will indicate (either through truth or lies) that the other is an Alg main. Now suppose that the Alg main will then indicate (either through truth or lies, depending on what they are pretending to main) that the Geo or Combo main is actually an Alg main.

Furthermore, suppose that for each pair of Alg mains, they will indicate (either through truth or lies, depending on what they are pretending to main), that the other person mains what they are pretending to main. (For example, given a pair of a fake Geo main A_G and a fake Combo main A_C, A_G will indicate through truth that A_C is a Combo main, and A_C will indicate through lies that A_G is a Geo main.)

Then it is impossible to tell the real Geo mains apart from the fake Geo mains, and same for Combo. It's possible that we have one Geo main, one Combo main, and the rest are Alg mains. It is also possible, however, based on the responses, that the fake Geo mains and Combo mains are real, and that the real Geo and Combo mains are actually Alg mains. □

Problem 2. Find the number of ordered quadruples of positive integers (a, b, c, d) satisfying $a! \cdot b! \cdot c! \cdot d! = 24!$.

Answer. $\boxed{28}$

Solution. We start by assuming WLOG that $a \geq b \geq c \geq d$. Notice that since the right side is divisible by 23, the left side must be as well. Thus, one of a, b, c, d must be at least 23 (so it must be 23 or 24).

Case 1: If $a = 24$, then $b! \cdot c! \cdot d! = 1$, so $b = c = d = 1$. Then

$$\{a, b, c, d\} = \{24, 1, 1, 1\},$$

and there are $\binom{4}{1} = 4$ ordered quadruples that result from this case.

Case 2: If $a = 23$, then $b! \cdot c! \cdot d! = 24$. Since 24 is divisible by 3, one of b, c, d must be either 3 or 4.
Case 2.1: If $b = 3$, then $c! \cdot d! = 4$, so $c = d = 2$. Then

$$\{a, b, c, d\} = \{23, 3, 2, 2\},$$

and there are $\frac{4!}{2!} = 12$ ordered quadruples that result from this case.
Case 2.2: If $b = 4$, then $c! \cdot d! = 1$, so $c = d = 1$. Then

$$\{a, b, c, d\} = \{23, 4, 1, 1\},$$

and there are another 12 ordered quadruples resulting from this case.

Adding up all the cases, we get $4 + 12 + 12 = \boxed{28}$ total ordered quadruples satisfying the condition. □

Problem 3. Let $R(x)$ be the remainder polynomial obtained when x^{2024} is divided by $x^2 - 3x + 2$. Find the value of $R(3)$.

Answer. $\boxed{2^{2025} - 1}$

Solution. Note that $x^2 - 3x + 2 = (x - 1)(x - 2)$. By the Division Theorem, we have that

$$x^{2024} = (x^2 - 3x + 2)Q(x) + R(x),$$

where $Q(x)$ is some polynomial, and $R(x)$ is of the form $ax + b$ (since $\deg(R) < \deg(x^2 - 3x + 2)$). Thus, we can write

$$x^{2024} = (x - 1)(x - 2)Q(x) + ax + b.$$

Then

$$1^{2024} = (1 - 1)(1 - 2)Q(1) + a + b = a + b,$$

and

$$2^{2024} = (2 - 1)(2 - 2)Q(2) + 2a + b = 2a + b.$$

Thus, we have the following system of equations:

$$\begin{cases} a + b = 1 \\ 2a + b = 2^{2024} \end{cases}$$

Subtracting these two equations, we get $a = 2^{2024} - 1$, so $b = 2 - 2^{2024}$. Thus,

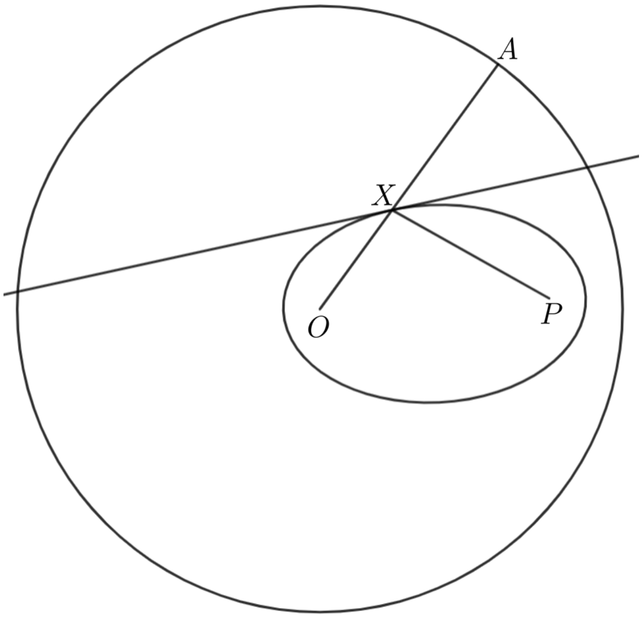
$$R(x) = (2^{2024} - 1)x + 2 - 2^{2024}$$

which means that

$$\begin{aligned} R(3) &= 3(2^{2024} - 1) + 2 - 2^{2024} \\ &= 3 \cdot 2^{2024} - 3 + 2 - 2^{2024} \\ &= 2 \cdot 2^{2024} - 1 \\ &= \boxed{2^{2025} - 1.} \end{aligned}$$

□

Problem 4. Let ω be a circle centered at a point O, and $P \neq O$ be a point inside ω. Let A be a point on ω, and X be the point on $\overline{OA}$ such that $XA = XP$. Show that as A moves along ω, X moves along an ellipse.



Solution. Notice that since $XA = XP$,

$$XO + XP = XO + XA = OA = r,$$

where r is the radius of ω. Thus, $XO + XP$ is constant, so by definition, X lies on an ellipse with foci O and P. □

Problem 5. Find all ordered positive integer 3-tuples (a, b, c) such that

$$\frac{1}{a+b-c} + \frac{1}{a+c-b} + \frac{1}{b+c-a} = \frac{1}{12}$$

$$\frac{c}{a+b-c} + \frac{b}{a+c-b} + \frac{a}{b+c-a} = 4.$$

Answer. $\boxed{(33,\ 44,\ 55),\ (30,\ 48,\ 54),\ \text{and their permutations.}}$

Solution. We begin by making the following substitution. Let

$$\begin{aligned} x &= b+c-a \\ y &= a+c-b \\ z &= a+b-c. \end{aligned}$$

Then the first equation simplifies to

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{12}.$$

Notice that

$$\begin{aligned} 2a &= y+z \\ 2b &= x+z \\ 2c &= x+y. \end{aligned}$$

Thus, the second equation becomes

$$\frac{x+y}{z} + \frac{x+z}{y} + \frac{y+z}{x} = 8.$$

Adding $\frac{x}{x} + \frac{y}{y} + \frac{z}{z} = 3$ to both sides, we obtain the equation

$$\frac{x+y+z}{z} + \frac{x+y+z}{y} + \frac{x+y+z}{x} = 11.$$

Thus,

$$11 = (x+y+z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = \frac{1}{12}(x+y+z).$$

So our original equations simplify to the following system:

$$\begin{cases} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{12} \\ x+y+z &= 132. \end{cases}$$

Lemma: $x, y, z > 0$. From before, we have

$$\begin{aligned} y+z &= 2a > 0 \\ x+z &= 2b > 0 \\ x+y &= 2c > 0 \end{aligned}$$

Hence, it cannot be the case that at least two of x, y, z are negative. Otherwise, some pair will sum to a negative number, contradicting the inequalities above.

Now, suppose for the sake of contradiction that exactly one of x, y, z is negative. WLOG, let it be x. Then,

$$\begin{aligned} x + z > 0 &\implies z > -x \\ x + y > 0 &\implies y > -x \end{aligned}$$

Now WLOG, let $y \le z$. This implies $2z \ge y + z = 132 - x > 132$, so $z > 66$. Using the first equation in the new system along with $y > -x$ and $z > 66$,

$$\begin{aligned} \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{12} \\ \implies \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{12} &= 0 \\ \implies \frac{1}{x} + \frac{1}{-x} + \frac{1}{66} - \frac{1}{12} > \frac{1}{x} + \frac{1}{y} + \frac{1}{z} - \frac{1}{12} &= 0 \\ \implies \frac{1}{66} - \frac{1}{12} &> 0 \end{aligned}$$

which is a contradiction, since $\frac{1}{66} - \frac{1}{12} < 0$. Hence, $x, y, z > 0$. From this lemma, we also obtain that $x, y, z > 12$, since $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} > \frac{1}{12}$ otherwise.

Motivated by Simon's Favorite Factoring Trick on two variables, we can multiply the first equation by $12xyz$ on both sides and move everything to one side to get the system

$$\begin{cases} xyz - 12xy - 12yz - 12xz = 0 \\ x + y + z - 132 = 0. \end{cases}$$

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This looks reminiscent of the expansion of $(x-12)(y-12)(z-12)$, so we expand:

$$\begin{aligned} (x-12)(y-12)(z-12) &= xyz - 12xy - 12yz - 12xz + 144x + 144y + 144z - 12^3 \\ &= 0 + 144(x + y + z - 132) + 144 \cdot 132 - 12^3 \\ &= 12^3 \cdot 11 - 12^3 \\ &= 10 \cdot 12^3. \end{aligned}$$

Now, we have the system

$$\begin{cases} (x-12)(y-12)(z-12) = 10 \cdot 12^3 \\ x + y + z = 132 \end{cases}$$

Consider the transformation of positive integer pairs $f : (x, y, z) \mapsto (x + 12, y + 12, z + 12)$, as we have shown $x, y, z > 12$. This simplifies the new system to

$$\begin{cases} xyz = 10 \cdot 12^3 \\ x + y + z = 96 \end{cases}$$

A prime p dividing both 96 and $10 \cdot 12^3$ cannot divide exactly two of x, y, z, because if WLOG $p \mid x, y$ and $p \nmid z$, then $p \mid 96 - x - y = z$, a contradiction. It also cannot be the case that p divides none of x, y, z, because then, $p \nmid xyz = 10 \cdot 12^3$, a contradiction. If p divides only one of x, y, z, say WLOG x, then all powers of that prime in $10 \cdot 12^3$ must divide x. For example, consider $p = 2$, which gives $128 \mid x$, impossible. Hence, $p = 2$ must divide all of x, y, z, so we apply another transformation of positive integer pairs $f : (x, y, z) \mapsto (2x, 2y, 2z)$. This gives

$$\begin{cases} xyz = 10 \cdot 6^3 \\ x + y + z = 48 \end{cases}$$

Using $p = 3$ this time, we get $27 \mid x$ WLOG or $3 \mid x, y, z$. If $27 \mid x$, then $x = 27$, which gives the system

$$\begin{cases} yz = 80 \\ y + z = 21 \end{cases}$$

Then, y and z are the zeroes of the quadratic $r^2 - 21r + 80$, which are 5 and 16. This means that the solutions in this case are $(27, 16, 5)$ and all of its permutations. Reverting through the transformations and substitutions, we get the ordered triple $(a, b, c) = (33, 44, 55)$ and all of its permutations.

If instead $3 \mid x, y, z$, another transformation $f : (x, y, z) \mapsto (3x, 3y, 3z)$ gives

$$\begin{cases} xyz = 10 \cdot 2^3 \\ x + y + z = 16 \end{cases}$$

We can go through this process again with $p = 2$. Either $x = 16$, which is impossible, or $2 \mid x, y, z$. Transforming again gives

$$\begin{cases} xyz = 10 \\ x + y + z = 8. \end{cases}$$

It is now clear that $(5, 2, 1)$ and its permutations are the only solutions. Reverting back gives $(a, b, c) = (30, 48, 54)$ and its permutations.

Hence, our final answer is $(33, 44, 55)$ and $(30, 48, 54)$ and all of their permutations.

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