

# HMMT November 2021

November 13, 2021

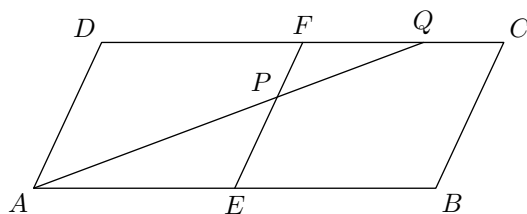
## Team Round

1. [20] Let  $ABCD$  be a parallelogram. Let  $E$  be the midpoint of  $AB$  and  $F$  be the midpoint of  $CD$ . Points  $P$  and  $Q$  are on segments  $EF$  and  $CF$ , respectively, such that  $A$ ,  $P$ , and  $Q$  are collinear. Given that  $EP = 5$ ,  $PF = 3$ , and  $QF = 12$ , find  $CQ$ .

*Proposed by: Daniel Zhu*

**Answer:**

**Solution:**



Triangles  $PFQ$  and  $PEA$  are similar, so  $AE = FQ \cdot \frac{PE}{PF} = 12 \cdot \frac{5}{3} = 20$ . Now,  $CQ = CF - QF = 20 - 12 = 8$ .

2. [25] Joey wrote a system of equations on a blackboard, where each of the equations was of the form  $a + b = c$  or  $a \cdot b = c$  for some variables or integers  $a, b, c$ . Then Sean came to the board and erased all of the plus signs and multiplication signs, so that the board reads:

$$x \quad z = 15$$

$$x \quad y = 12$$

$$x \quad x = 36$$

If  $x, y, z$  are integer solutions to the original system, find the sum of all possible values of  $100x + 10y + z$ .

*Proposed by: David Vulakh*

**Answer:**

**Solution:** The bottom line gives  $x = -6$ ,  $x = 6$  or  $x = 18$ . If  $x = -6$ ,  $y$  can be  $-2$  or  $18$  and  $z$  must be  $21$ , so the possible values for  $100x + 10y + z$  are  $-599$  and  $-399$ . If  $x = 6$ ,  $y$  can be  $2$  or  $6$  and  $z$  must be  $9$ , so the possible values are  $629$  and  $669$ . If  $x = 18$ ,  $y$  must be  $-6$  and  $z$  must be  $-3$ , so the only possible value is  $1737$ . The total sum is  $2037$ .

3. [30] Suppose  $m$  and  $n$  are positive integers for which

- the sum of the first  $m$  multiples of  $n$  is  $120$ , and
- the sum of the first  $m^3$  multiples of  $n^3$  is  $4032000$ .

Determine the sum of the first  $m^2$  multiples of  $n^2$ .

*Proposed by: Sean Li*

**Answer:**

**Solution:** For any positive integers  $a$  and  $b$ , the sum of the first  $a$  multiples of  $b$  is  $b + 2b + \dots + ab = b(1 + 2 + \dots + a) = \frac{a(a+1)b}{2}$ . Thus, the conditions imply  $m(m+1)n = 240$  and  $m^3(m^3+1)n^3 = 8064000$ , whence

$$\frac{(m+1)^3}{m^3+1} = \frac{(m(m+1)n)^3}{m^3(m^3+1)n^3} = \frac{240^3}{8064000} = \frac{12}{7}.$$

Thus, we have  $7(m+1)^2 = 12(m^2 - m + 1)$  or  $5m^2 - 26m + 5 = 0$ , so  $m = 5$  and therefore  $n = 8$ . The answer is  $\frac{m^2(m^2+1)}{2}n^2 = 20800$ .

4. [35] Find the number of 10-digit numbers  $\overline{a_1a_2 \dots a_{10}}$  which are multiples of 11 such that the digits are non-increasing from left to right, i.e.  $a_i \geq a_{i+1}$  for each  $1 \leq i \leq 9$ .

*Proposed by: Sheldon Kieren Tan*

**Answer:**

**Solution:** It is well known that  $\overline{a_1a_2 \dots a_{10}}$  is divisible by 11 if and only if  $S = a_1 - a_2 + a_3 - \dots - a_{10}$  is. By the non-increasing condition, we deduce that

$$S = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_9 - a_{10}) \geq 0.$$

Also,

$$S = a_1 - (a_2 - a_3) - \dots - (a_8 - a_9) - a_{10} \leq a_1 \leq 9.$$

Therefore,  $S = 0$ , our number must be of the form  $\overline{abbccddde}$ . Since all numbers of this form work ( $\overline{abbccddde} = 11 \cdot \overline{a0b0c0d0e}$ ), it suffices to find the number of tuples  $(a, b, c, d, e)$  so that  $9 \geq a \geq b \geq c \geq d \geq e \geq 0$  and  $a > 0$ . The number of tuples satisfying the first condition is  $\binom{14}{5} = 14 \cdot 13 \cdot 11 = 2002$ . To account for the second condition, we subtract one tuple (all zeroes), yielding a final answer of 2001.

5. [40] How many ways are there to place 31 knights in the cells of an  $8 \times 8$  unit grid so that no two attack one another?

(A knight attacks another knight if the distance between the centers of their cells is exactly  $\sqrt{5}$ .)

*Proposed by: Frederick Zhao*

**Answer:**

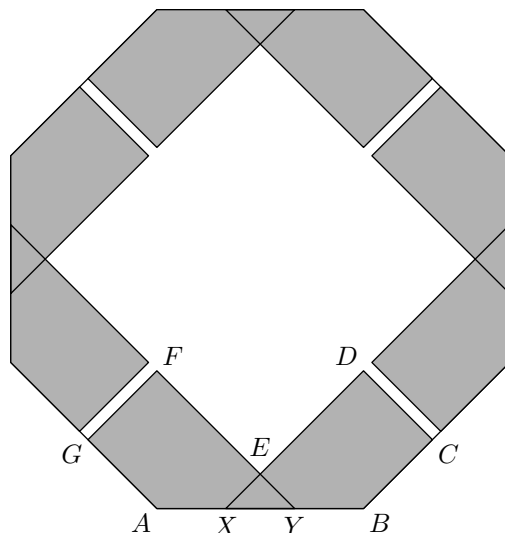
**Solution:** Consider coloring the squares of the chessboard so that 32 are black and 32 are white, and no two squares of the same color share a side. Then a knight in a square of one color only attacks squares of the opposite color. Any arrangement of knights in which all 31 are placed on the same color therefore works: there are 64 such arrangements (one for each square, in which that square is empty and the others of the same color are occupied). Also, if a knight is placed in a corner, it only attacks two squares. Therefore, for each corner, it is possible to place a knight in one corner and in all squares of the opposite color except the two attacked by the corner knight. This gives 68 total arrangements. one can prove that no others are possible.

6. [40] The *taxicab distance* between points  $(x_1, y_1)$  and  $(x_2, y_2)$  is  $|x_2 - x_1| + |y_2 - y_1|$ . A regular octagon is positioned in the  $xy$  plane so that one of its sides has endpoints  $(0, 0)$  and  $(1, 0)$ . Let  $S$  be the set of all points inside the octagon whose taxicab distance from some octagon vertex is at most  $\frac{2}{3}$ . The area of  $S$  can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .

*Proposed by: David Vulakh*

**Answer:**

**Solution:**



In the taxicab metric, the set of points that lie at most  $d$  units away from some fixed point  $P$  form a square centered at  $P$  with vertices at a distance of  $d$  from  $P$  in directions parallel to the axes. The diagram above depicts the intersection of an octagon with eight such squares for  $d = \frac{2}{3}$  centered at its vertices. (Note that since  $\sqrt{2} > \frac{2}{3} \cdot 2$ , the squares centered at adjacent vertices that are diagonal from each other do not intersect.) The area of the entire shaded region is  $4[ABCDEFG] = 4(2([AFG] + [AYF]) - [EXY])$ , which is easy to evaluate since  $AFG$ ,  $AYF$ , and  $EXY$  are all 45-45-90-degree triangles. Since  $AF = \frac{2}{3}$ ,  $GF = \frac{\sqrt{2}}{3}$ , and  $EX = \frac{1}{3\sqrt{2}}$ , the desired area is  $4\left(\frac{2}{9} + \frac{4}{9} - \frac{1}{36}\right) = \frac{23}{9}$ .

7. [45] Let  $f(x) = x^3 + 3x - 1$  have roots  $a, b, c$ . Given that

$$\frac{1}{a^3 + b^3} + \frac{1}{b^3 + c^3} + \frac{1}{c^3 + a^3}$$

can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ , find  $100m + n$ .

*Proposed by: Milan Haiman*

**Answer:**

**Solution:** We know that  $a^3 = -3a + 1$  and similarly for  $b, c$ , so

$$\frac{1}{a^3 + b^3} = \frac{1}{2 - 3a - 3b} = \frac{1}{2 + 3c} = \frac{1}{3(2/3 + c)}.$$

Now,

$$f(x - 2/3) = x^3 - 2x^2 + \frac{13}{3}x - \frac{89}{27}$$

has roots  $a + 2/3$ ,  $b + 2/3$ , and  $c + 2/3$ . Thus the answer is, by Vieta's formulas,

$$\frac{1}{3} \frac{(a + 2/3)(b + 2/3) + (a + 2/3)(c + 2/3) + (b + 2/3)(c + 2/3)}{(a + 2/3)(b + 2/3)(c + 2/3)} = \frac{13/3}{3 \cdot 89/27} = \frac{39}{89}.$$

8. [50] Paul and Sara are playing a game with integers on a whiteboard, with Paul going first. When it is Paul's turn, he can pick any two integers on the board and replace them with their product; when

it is Sara's turn, she can pick any two integers on the board and replace them with their sum. Play continues until exactly one integer remains on the board. Paul wins if that integer is odd, and Sara wins if it is even.

Initially, there are 2021 integers on the board, each one sampled uniformly at random from the set  $\{0, 1, 2, 3, \dots, 2021\}$ . Assuming both players play optimally, the probability that Paul wins is  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find the remainder when  $m + n$  is divided by 1000.

*Proposed by: David Vulakh*

**Answer:** 383

**Solution:** We claim that Paul wins if and only if there are exactly 1 or 2 odd integers on the board at the start. Assuming this, the answer is  $\frac{2021 + \binom{2021}{2}}{2^{2021}}$ . Since the numerator is odd, this fraction is reduced. Now,  $m + n \equiv 2^{2021} + 21 + 2021 \cdot 1010 \equiv 231 + 2^{2021} \equiv 231 + 2^{21} \equiv 231 + 2 \cdot 1024^2 \equiv 231 + 2 \cdot 576 \equiv 383$ .

Now, observe that only the parity of the integers matters, so we work mod 2, replacing all odd integers with ones and even integers with zeros. Also, note that on Paul's turn, there are always an odd number of numbers on the board, and vice versa.

If the number of ones on the board ever becomes 1, Paul can win, since Sara cannot change the number of ones on the board, while Paul can replace 2 zeros with 1 zero, since Paul will always be given at least 3 numbers on his turn. Moreover, if at the beginning there are 2 ones, Paul can replace them with 1 one and win in the same manner. Obviously, if at any point the board only contains zeroes, Sara wins.

Now suppose the number of ones on the board is initially at least 3. Call a state *good* if there are at least 3 ones and at least 1 zero. We now make the following claims:

**Claim.** If Paul ever acts on a good state so that the result is no longer good, Sara can force a win.

*Proof.* Paul cannot erase all the zeros from the board. Also, Paul can decrease the number of ones on the board by at most 1. Therefore, the only way this can happen is if, as a result of Paul's move, the number of ones drops from 3 to 2. However, in the case, Sara can replace the 2 ones with a zero on her next turn, making the board contain all zeros, guaranteeing a Sara victory. ■

**Claim.** If the current state of the game is good, Sara can make a move that results in a good state, with the exception of 1110, in which case Sara can win anyway.

*Proof.* If there are at least 2 zeros, Sara can act on those. If there are at least 5 ones, Sara can replace 2 ones with a zero. If none of these are true, then there must be at most 1 zero and at most 4 ones. Since Sara will always have an even number of numbers on the board on her turn, the state must be 1110. In this case, she may replace a one and a zero with a one, giving Bob the state 111. The only move for Bob is to change the state to 11, after which Alice wins following her only move. ■

As a result of these claims, if the state of the board ever becomes good, Sara can force a win. Now, if at the beginning there are at least 3 ones, the state is either good already. Otherwise, the state consists of 2021 ones. In the latter case, Paul must change the state to 2020 ones, after which Sara can replace 2 ones with a zero, making the state 2018 ones and 1 zero. Since  $2018 \geq 3$ , the state is now good and therefore Sara can force a win.

9. [55] Let  $N$  be the smallest positive integer for which

$$x^2 + x + 1 \text{ divides } 166 - \sum_{d|N, d>0} x^d.$$

Find the remainder when  $N$  is divided by 1000.

*Proposed by: Joseph Heerens*

**Answer:** 672

**Solution:** Let  $\omega = e^{2\pi i/3}$ . The condition is equivalent to

$$166 = \sum_{d|N, d>0} \omega^d.$$

Let's write  $N = 3^d n$  where  $n$  is not divisible by 3. If all primes dividing  $n$  are 1 mod 3, then  $N$  has a positive number of factors that are 1 mod 3 and none that are 2 mod 3, so  $\sum_{d|N, d>0} \omega^d$  has nonzero imaginary part. Therefore  $n$  is divisible by some prime that is 2 mod 3. In this case, the divisors of  $n$  are equally likely to be 1 or 2 mod 3, so the sum is

$$-\frac{1}{2}\tau(n) + d\tau(n) = \frac{2d-1}{2}\tau(n).$$

Now,  $2 \cdot 166 = 2^2 \cdot 83$  and 83 is prime, so we must either have  $d = 42$ , which forces  $\tau(n) = 4$ , or  $d = 1$ , which forces  $\tau(n) = 332$ . The first case yields a lower value of  $N$ , namely  $3^{42}2^3$ .

Now let's try to compute this mod 1000. This is clearly divisible by 8. Modulo 125,  $3^5 = 243 \equiv -7$ , so  $3^{20} \equiv 2401 \equiv 26$  and  $3^{40} \equiv 676 \equiv 51$ . Therefore  $3^{42}2^3 \equiv 72 \cdot 51 = 3672 \pmod{125}$ . Since 672 is divisible by 8, this is our answer.

10. **[60]** Three faces  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{Z}$  of a unit cube share a common vertex. Suppose the projections of  $\mathcal{X}$ ,  $\mathcal{Y}$ ,  $\mathcal{Z}$  onto a fixed plane  $\mathcal{P}$  have areas  $x$ ,  $y$ ,  $z$ , respectively. If  $x : y : z = 6 : 10 : 15$ , then  $x + y + z$  can be written as  $\frac{m}{n}$ , where  $m, n$  are positive integers and  $\gcd(m, n) = 1$ . Find  $100m + n$ .

*Proposed by: Sean Li*

**Answer:**

**Solution:** Introduce coordinates so that  $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$  are normal to  $(1, 0, 0)$ ,  $(0, 1, 0)$ , and  $(0, 0, 1)$ , respectively. Also, suppose that  $\mathcal{P}$  is normal to unit vector  $(\alpha, \beta, \gamma)$  with  $\alpha, \beta, \gamma \geq 0$ .

Since the area of  $\mathcal{X}$  is 1, the area of its projection is the absolute value of the cosine of the angle between  $\mathcal{X}$  and  $\mathcal{P}$ , which is  $|(1, 0, 0) \cdot (\alpha, \beta, \gamma)| = \alpha$ . (For parallelograms it suffices to use trigonometry, but this is also true for any shape projected onto a plane. One way to see this is to split the shape into small parallelograms.) Similarly,  $y = \beta$  and  $z = \gamma$ . Therefore  $x^2 + y^2 + z^2 = 1$ , from which it is not hard to calculate that  $(x, y, z) = (6/19, 10/19, 15/19)$ . Therefore  $x + y + z = 31/19$ .