Stuyvesant Team Contest: Solutions

Winter 2021

Problem 1. **[6]** A point is chosen inside a unit square at random. What is the probability that the distance from this point to the closest side is at most $\frac{1}{2}$?

Answer. $\boxed{1}$ *Proposed by Rishabh Das*

Solution. No matter what point is chosen, it is always at most $\frac{1}{2}$ away from a side, so the probability is 1. □

Problem 2. **[6]** Two positive integers x and y sum up to 41, with $x < y$. Find the maximum possible value of $100x + y$.

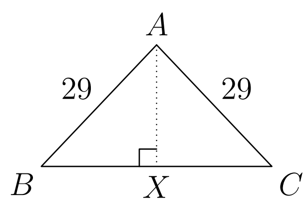
Answer. $\boxed{2021}$ *Proposed by Rishabh Das*

Solution. Since x is being multiplied by 100, we would like to maximize x. This occurs when $x = 20$ and $y = 21$, and this makes $100x + y = 2021$. □

Problem 3. **[7]** Triangle ABC has $AB = AC = 29$ and perimeter 98. What is the area of the triangle?

Answer. $\boxed{420}$ *Proposed by Rishabh Das*

Solution. Note that this triangle is isosceles, so the foot from A to BC is the midpoint of BC, X.



Since the perimeter is 98, $BC = 98 - AB - AC = 98 - 2 \cdot 29 = 40$. Since X is the midpoint of BC, $BX = XC = 20$. By the Pythagorean Theorem, $AX = \sqrt{29^2 - 20^2} = 21$. Then the area of the triangle is

$$\frac{40 \cdot 21}{2} = 420.$$

□

Problem 4. **[7]** Mr. Kats writes the letters

STCSTC

on a board. Mr. Sterr sees this, and erases three of the letters. How many ways can he do this so that the three remaining letters read "STC" in that order?

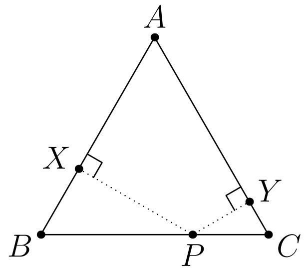
Answer. $\boxed{4}$ *Proposed by Rishabh Das*

Solution. If the last "S" remains on the board, then there is only one way to choose the "T" and "C". Otherwise, the first "S" is selected. If the last "T" is selected, there is only one way to choose the "C". Otherwise, the first "T" is selected, and then there are 2 ways to pick the "C". In total, there are 4 ways to pick the letters. □

Problem 5. **[8]** Equilateral triangle $\triangle ABC$ has side length 2. A point P is selected on segment BC such that $\frac{BP}{PC} = \frac{20}{21}$. Points X and Y are selected on AB and AC respectively such that $PX \perp AB$ and $PY \perp AC$. Compute $AX + AY$.

Answer. $\boxed{3}$ *Proposed by Rishabh Das*

Solution. The answer is 3, invariant of which point P on BC is picked.



Note that $BX = \frac{BP}{2}$ since BXP is a $30-60-90$ triangle. Similarly, $CY = \frac{CP}{2}$. Thus,

$$AX + AY = (AB + AC) - (BX + CY) = (AB + AC) - \frac{BC}{2} = 2 + 2 - 1 = 3.$$

□

Problem 6. **[8]** What's the largest positive integer that can't be written as the sum of two composite positive integers?

Answer. $\boxed{11}$ *Proposed by Rishabh Das*

Solution. Any even number $2k$ at least 8 can be written as $(2k-4)+4$, while any odd numbers $2k+1$ at least 13 can be written as $(2k-8)+9$. Thus, the largest number still in contention is 11, which can be written as $1+10, 2+9, 3+8, 4+7$, or $5+6$, none of which are two composite numbers. □

Problem 7. **[9]** The sequence a, b, c is an increasing arithmetic sequence of positive integers and $a, b, c+1$ is an increasing geometric sequence of positive integers. Given that b is at least 2021, find the smallest possible value of b.

Answer. $\boxed{2070}$ *Proposed by Rishabh Das*

Solution. Let $a = b - x$ and $c = a + x$. The geometric sequence condition says

$$a(c+1) = b^2 \implies (b-x)(b+x+1) = b^2.$$

Expanding, this gives $b^2 - x^2 + b - x = b^2$, or $b = x^2 + x$. Thus, all such triples (a, b, c) are of the form

$$(x^2, x^2 + x, x^2 + 2x).$$

We want $x^2 + x \geq 2021$, which first occurs when $x = 45$, which makes $b = 2070$. □

Problem 8. **[9]** Compute

$$\frac{(1 + 2020^2 + 2021^2)^2}{2020^2 + 2021^2 + (2020 \cdot 2021)^2}.$$

Answer. $\boxed{4}$ *Proposed by Rishabh Das*

Solution. Let $2020 = n$. The numerator is equal to

$$(1 + n^2 + (n+1)^2)^2 = (2n^2 + 2n + 2)^2 = 4(n^2 + n + 1)^2,$$

while the denominator is equal to

$$n^2 + (n+1)^2 + n^2(n+1)^2 = n^4 + 2n^3 + 3n^2 + 2n + 1 = (n^2 + n + 1)^2.$$

Thus, the fraction is equal to 4. □

Problem 9. **[10]** The numbers $15, 25, 35, 45, 55, 65, 75, 85$, and 95 are written on pieces of paper in a hat. Mr. Cocoros grabs two distinct pieces of paper from the hat, and computes the product of the numbers. What is the probability that the tens digit of the result is 2?

Answer. $\boxed{\frac{4}{9}}$ *Proposed by Rishabh Das*

Solution. Suppose the numbers $10a+5$ and $10b+5$ are selected. Then the product is

$$(10a+5)(10b+5) = 100ab + 50(a+b) + 25.$$

The last two digits of this is the same as the last two digits of $50(a+b)+25$, which is 25 when $a+b$ is even and 75 when $a+b$ is odd.
The numbers in the hat are $10x+5$ for $1 \leq x \leq 9$. There are 4 even values and 5 odd values of x. Thus, there are $4 \cdot 3 + 5 \cdot 4 = 32$ ways to select the two numbers such that the sum of their tens digits is even. There are $9 \cdot 8 = 72$ total ways to select two numbers, so the probability is $\frac{32}{72} = \frac{4}{9}$. □

Problem 10. **[Up to 10]** There are N teams competing in STC. Pick an integer X between 1 and 10, inclusive. Let k teams (including your team) pick X. If $k > \frac{N}{10}$, you will receive 0 points. Otherwise, you will receive X points.

Answer. $\boxed{\text{N.A.}}$ *Proposed by Rishabh Das*

Here are the results:

	1	2	3	4	5	6	7	8	9	10
# Of Teams	0	1	1	5	2	4	2	2	2	9
Points Earned	1	2	3	0	5	0	7	8	9	0

Problem 11. **[11]** Mr. Cocoros chooses a two-element subset of S at random. What is the expected value of the positive difference between the two elements?

Answer. $\boxed{7}$ *Proposed by Srinath Mahankali*

Solution. There are $20-k$ ways to choose the two elements such that their difference is k: the smaller number can be anything from 1 to $20-k$. There are also $\binom{20}{2}$ ways to pick S. By the definition of expected value, the answer is

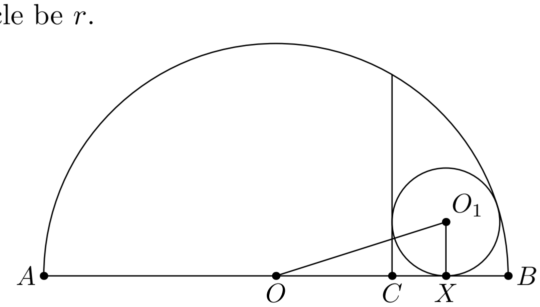
$$\sum_{k=1}^{20} k \cdot \frac{20-k}{190} = \frac{1}{190}\sum_{k=1}^{20} 20k - k^2 = \frac{1}{190}\left(20 \cdot \frac{20 \cdot 21}{2} - \frac{20 \cdot 21 \cdot 41}{6}\right) = \frac{1}{190}\left(\frac{20 \cdot 21 \cdot 19}{6}\right) = 7.$$

□

Problem 12. **[11]** Let AB be the diameter of a semicircle, ω, with $AB = 4$. Let O be the midpoint of AB, and C be the midpoint of OB. The perpendicular to AB through C intersects ω again at D. A circle is drawn tangent to segments CD and CB, and to arc BD of ω. What is the radius of this circle?

Answer. $\boxed{2\sqrt{3}-3}$ *Proposed by Rishabh Das*

Solution. Let the radius of the circle be r.



The distance OX is equal to $OX + CX = 1 + r$. The distance O_1X is just equal to r, and the distance OO_1 is equal to $2 - r$. By the Pythagorean Theorem on $\triangle OXO_1$, we see

$$r^2 + (1+r)^2 = (2-r)^2 \implies r^2 + r^2 + 2r + 1 = r^2 - 4r + 4 \implies r^2 + 6r - 3 = 0.$$

The solutions to this quadratic are $r = -3 \pm 2\sqrt{3}$; since r must be positive, it is equal to $2\sqrt{3} - 3$. □

Problem 13. **[12]** A subset S of $\{1, 3, 5, \ldots 2021\}$ is chosen such that if a and b are any distinct elements of S, then a does not divide b. What is the maximum possible size of S?

Answer. $\boxed{674}$ *Proposed by Rishabh Das*

Solution. A construction for $|S| = 674$ is to take $S = \{675, 677, 679, \ldots 2021\}$. The ratio between any two terms in this set is less than 3, and since all terms of S are odd, the ratio cannot equal 2. Thus, this set S works.
Now we must show that 674 is indeed the maximum. To do this, define

$$S_k = \{k \cdot 3^r | r \geq 0\}$$

for $3 \nmid k$. It is clear that we may take at most one value from each set, as otherwise the ratio of the two would be a power of 3. There are 674 elements of $\{1, 3, 5, \ldots 2021\}$ that aren't multiples of 3, and since we can select at most one number from each set corresponding to that element, the size must be at most 674. □

Problem 14. **[12]** The equation $x^2 - 2x + c = 0$ has roots $\tan\theta$ and $\sec\theta$ for some $\theta \in (0°, 90°)$. Compute c.

Answer. $\boxed{\dfrac{15}{16}}$ *Proposed by Rishabh Das*

Solution. The sum of the roots is 2, so let $\tan\theta = 1 - t$ and $\sec\theta = 1 + t$. Using the property $\sec^2 x - \tan^2 x = 1$, this means

$$(1+t)^2 - (1-t)^2 = 1 \implies 4t = 1,$$

so $t = \frac{1}{4}$. Then $c = (1+t)(1-t) = 1 - t^2 = \frac{15}{16}$. □

Problem 15. **[13]** For $n \geq 2$, define $f(n)$ as the largest proper divisor d of n for which the number of positive integer divisors of d is maximized. For which k is

$$\underbrace{f(f(f(\cdots f(}_{k\ f\text{s}}10!)\cdots))) = 1,$$

where f is iterated k times?
Note: A *proper divisor* of n is a divisor of n that is not n.

Answer. $\boxed{15}$ *Proposed by Rishabh Das*

Solution. Let n be the product of m primes. Then note that $f(n)$ will be the product of $m-1$ primes, since losing more primes means losing more divisors. We may continue in this manner, to get $f^k(n)$ is the product of $m - k$ primes, where f^k represents the function f iterated k times. This means that $f^m(n)$ is the product of 0 primes, meaning that it is 1. Thus, if 10! is the product of m primes then the answer is m.
We can write

$$10! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = (2) \cdot (3) \cdot (2^2) \cdot (5) \cdot (2 \cdot 3) \cdot (7) \cdot (2^3) \cdot (3^2) \cdot (2 \cdot 5) = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7,$$

so 10! is the product of $8 + 4 + 2 + 1 = 15$ primes. □

Problem 16. **[13]** A square is centered at the origin. Every second, Mr. Sterr selects a vertex of the square at random, and the square is reflected over the selected vertex. What is the probability that after 4 seconds, the square is in its original position?

Answer. $\boxed{\dfrac{9}{64}}$ *Proposed by Rishabh Das*

Solution. Without loss of generality let the square have vertices $\left(\pm\frac{1}{2},0\right),\left(0,\pm\frac{1}{2}\right)$. We track the center of the square, which needs to return back to the origin. Each move, it will make one of the moves $(\pm 1,0)$ or $(0,\pm 1)$. Call these moves x-moves and y-moves, respectively.
If we make 4 x-moves, then there must be 2 $(+1,0)$s and 2 $(-1,0)$s, resulting in $\binom{4}{2}=6$ possibilities. Similarly, if there are 4 y-moves then there are 6 possibilities.
If we make 2 x-moves and 2 y-moves, then we must have 1 of every move. This means we have $4!=24$ possibilities. Overall, there are $6\cdot 2+24=36$ possibilities. There are $4^4=256$ different sequences of moves, so the probability is $\frac{36}{256}=\frac{9}{64}$. □

Problem 17. [**14**] A sphere goes through the point $O=(0,0,0)$ and is tangent to the plane $x+y+z=1$ at the point $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. Suppose this sphere intersects the $x,y,$ and z axes at the points $X,Y,$ and Z respectively, where $O\neq X,Y,Z$. Compute the volume of tetrahedron $OXYZ$.

Answer. $\boxed{\dfrac{1}{162}}$ *Proposed by Rishabh Das*

Solution. It is possible (and not too bad) to find the equation for the sphere and plug in $y=z=0$ to get x, but we present another solution.
Let $A=(1,0,0), B=(0,1,0)$, and $C=(0,0,1)$. Also let $T=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$.
By power of a point from A to the sphere, we see

$$AT^2=AO\times AX.$$

We may compute AT^2 as

$$\left(1-\frac{1}{3}\right)^2+\left(0-\frac{1}{3}\right)^2+\left(0-\frac{1}{3}\right)^2=\frac{2}{3}$$

and AO is just 1. This means that $AX=\frac{2}{3}$, so $OX=\frac{1}{3}$. Similarly, we find $OY=OZ=\frac{1}{3}$. Then the volume of tetrahedron $OXYZ$ is

$$\frac{\left(\frac{1}{3}\right)^3}{6}=\frac{1}{162}.$$

□

Problem 18. [**14**] If $x>1$ and

$$\log_5\left(\log_5 x\right)+4\log_x\left(\log_x 5\right)=0,$$

find the sum of all possible values of x.

Answer. $\boxed{630}$ *Proposed by Rishabh Das*

Solution. Let $x=5^{5^y}$. Then $\log_5(\log_5(x))=y$.
Now $\log_x(5)=\frac{1}{\log_5(x)}=\frac{1}{5^y}$. Then

$$\log_x(\log_x(5))=\log_x\left(\frac{1}{5^y}\right)=-\log_x(5^y)=-\frac{1}{\log_{5^y}(5^{5^y})}=-\frac{1}{\frac{5^y}{y}}=-\frac{y}{5^y}.$$

Thus, the equation means $y-\frac{4y}{5^y}=0$, or

$$y(5^y-4)=0.$$

This means either $y=0$ or $5^y=4$. The first one means $x=5$ and the second one means $x=5^{5^y}=5^4=625$. Their sum is 630. □

Problem 19. [**15**] Call a rectangle n-*special* if it has integer dimensions, and the sum of its area and perimeter is equal to n. Compute the sum of the smallest two values of n for which there are exactly 5 noncongruent n-special rectangles.

Answer. $\boxed{499}$ *Proposed by Rishabh Das*

Solution. Let the dimensions of the rectangle be $a \times b$, and assume without loss of generality that $a \leq b$. The condition says

$$ab + 2(a+b) = n \implies (a+2)(b+2) = n + 4 = m$$

via Simon's Favorite Factoring Trick. Let $x = a + 2 \geq 3$ and $y = b + 2 \geq 3$. Then there must be 5 solutions to $xy = n$, where $3 \leq x \leq y$. There are four cases on the value of m.

If m is an even square, then it must have 13 divisors, since then there are 7 unordered pairs of integers multiplying to m, but 2 of them don't count. The smallest such number is $2^{12} = 4096$.

If m is even but not a square, then it must have 14 divisors by similar reasoning. The smallest such number is $2^6 \cdot 3 = 192$, and the next smallest number is $2^6 \cdot 6 = 320$.

If m is an odd square, then it must have 11 divisors, since there are 6 unordered pairs multiplying to m, but 1 of them doesn't count. The smallest such number is $3^{10} = 59049$.

If m is odd but not a square, then it must have 12 divisors by similar reasoning. The smallest such number is $3^2 \cdot 5 \cdot 7 = 315$.
Thus, the smallest two values of m are 192 and 315. Thus, the smallest two values of n are 188 and 311, which have a sum of 499. □

Problem 20. [Up to 64] Welcome to **USAYNO**!
Instructions: Submit a string of 6 *letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive* 2^n *points for n correct answers, but you will receive* 0 *if any of the questions you choose to answer is answered incorrectly. Note that this means if you submit "XXXXXX" you will get one point.*
(1) A *perfect power* is a number of the form n^k for an integer $k \geq 2$ and a positive integer n. There are infinitely many perfect powers that are also palindromes.
(2) If $\tau(n)$ is equal to the number of positive divisors of n, then

$$\sum_{k=1}^{\infty} \frac{(\tau(k))^{2021}}{k^2}$$

diverges.
(3) There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) + f(y) \leq x - y$$

for all $x, y \in \mathbb{R}$.
(4) There exists a set $\mathcal{S}$ of squares, each with finite, positive area in $\mathbb{R}^2$ such that every point in $\mathbb{R}^2$ is part of exactly one square in $\mathcal{S}$. (Here a square does not consist of its interior, but only its border.)
(5) A particle on the number line starts at 0. Every second, it either moves forward by 1 or backwards by 1 at random. Let p_n be the probability that after n seconds, the absolute value of the position of the particle is at most $n^{2020/2021}$. Then

$$\lim_{n \longrightarrow \infty} p_n = 0.$$

(6) An $n \times n$ square is tiled with L-shaped trinominoes. Then 6 must divide n. (An L-shaped trinomino is a 2×2 square with one of the 1×1 corners cut out.)

Answer. TFTFFF *Proposed by Akash Das, Rishabh Das, and Srinath Mahankali*

Solution. (1) and (3) are true, while the rest are false.
(1) This is true; as a construction, take $(10^k + 1)^2$ for any k.

(2) This is false. We may separate by primes p, since $\frac{(\tau(x))^{2021}}{x^2}$ is a multiplicative function. Thus, the sum is equal to

$$\prod_p \left(\sum_{k=0}^{\infty} \frac{(k+1)^{2021}}{p^{2k}}\right).$$

Let $S_n(x) = \sum_{k=0}^{\infty} \frac{(k+1)^n}{x^{2k}}$. We can show by induction on n that

$$S_n(x) = \frac{x^{2n+2} + Q_n(x)}{(x^2-1)^{n+1}}$$

for some polynomial $Q_n(x)$ with degree at most $2n$. The base case of $n = 0$ is just summing a geometric series, while for larger n look at $x^2 S_{n+1}(x) - S_{n+1}(x)$ and write it as a sum of x^2 and $S_m(x)$ for $m \le n$.

Now our product is

$$\prod_p \left(\frac{p^{4044} + Q_{2021}(p)}{(p^2-1)^{2022}}\right).$$

Note that $Q_{2021}(p) \le cp^{4022}$ for some constant c for all primes p. Thus, we may upper-bound our product as

$$\left[\prod_{k=0}^{\infty} \frac{p^2}{p^2-1}\right]^{2022} \left[\prod_{k=0}^{\infty} 1 + \frac{c}{p^2}\right].$$

The value $\frac{p^2}{p^2-1}$ is equal to $\frac{1}{1-\frac{1}{p^2}} = 1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots$, so the first product actually expands to $\sum_{t=1}^{\infty} \frac{1}{t^2}$, which is well-known to converge.

Now we show the other term in the product converges as well. Note that

$$1 + \frac{c}{p^2} \le \left(1 + \frac{1}{p^2}\right)^c \le \left(1 + \frac{1}{p^2} + \frac{1}{p^4} + \cdots\right)^c,$$

so the product is upper-bounded by to $\left[\sum_{t=1}^{\infty} \frac{1}{t^2}\right]^c$, which converges for similar reasons to the other term. Thus, the entire summation converges.

(3) This is true. Two possible constructions are $-|x|$ and $-\frac{1}{4} - x^2$.

(4) This is false. We show that the interior of a single square X_1 in $\mathcal{S}$ cannot be covered, which is enough.

Note that for any square X in $\mathcal{S}$, there must exist a square Y inside of X such that the side length of Y is half that of X; if not, the center of X could not be covered.
Define a sequence of squares $X_1, X_2, X_3, \dots$ in $\mathcal{S}$ such that X_{i+1} is inside of X_i and has at most half the side length of that of X_i. Let P_i be the center of X_i for all i.
The sequence $P_1, P_2, P_3, \dots$ converges to a point P. Moreover, the square X_i get arbitrarily close to P. Thus, any square going through P must intersect some X_i, giving a contradiction.

(5) This is false. Let $2020/2021 = r > \frac{1}{2}$. Also replace n with $2n$.
First we compute the expected value of the absolute value of the position. Note that when the particle is not on 0, the expected value of its increase is 0, while if it is then the expected increase is 1. The expected value is thus the expected number of times the particle moves from 0. This is

$$E = \sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{4^k}.$$

A lower bound for this value would be

$$p_{2n} \cdot 0 + (1-p_{2n}) \cdot (2n)^r = (1-p_{2n}) \cdot (2n)^r \le E \implies p_{2n} \ge \frac{(2n)^r - E}{(2n)^r} = 1 - \frac{E}{(2n)^r} = 1 - c\frac{E}{n^r}$$

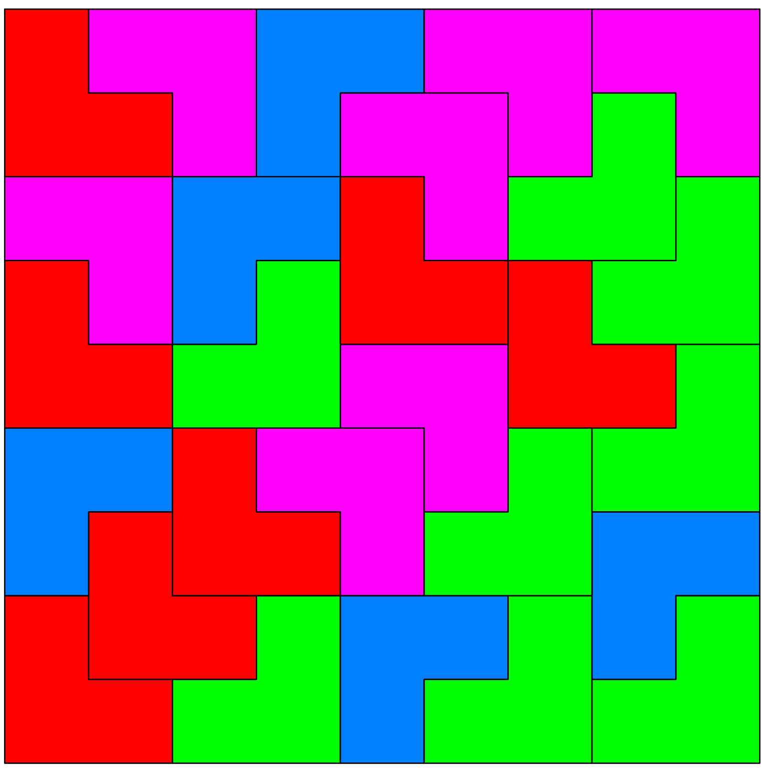
where $c = 2^{-r}$. We show that $\frac{E}{n^r}$ can get arbitrarily close to 0.

We can see that $\frac{\binom{2k}{k}}{4^k}$ is $O(1/\sqrt{k})$ by Stirling's Approximation. Thus,

$$E \le 1 + d\sum_{k=1}^{n-1} \frac{1}{\sqrt{k}}$$

for some constant d. Also note that the additional $+1$ and the constant d will not matter when we divide by n^r, so we discard them. The sum $\displaystyle\sum_{k=1}^{n-1} \frac{1}{\sqrt{k}}$ can be upper-bounded by $s\sqrt{n}$ for some constant s for all n by integrating $\frac{1}{\sqrt{x}}$ from $x = 0$ to $n-1$. Thus, $\frac{E}{n^r} \le \frac{s}{n^{r-0.5}}$, which approaches 0 as $r > 0.5$. Thus, p_n approaches 1, and not 0.

(6) This is false. Here is a construction on a 9×9 board.



After this 9×9 is constructed, we can actually construct any $(6k+3) \times (6k+3)$ board with $k \ge 1$ by taking this board, two $9 \times (6k-6)$ boards, and a $(6k-6) \times (6k-6)$ board, the latter three filled with 2×3 rectangles. Thus, an $n \times n$ board can be filled with such pieces if and only if $3 \mid n$ and $n \ne 3$. (The case of $n = 3$ can be checked manually to not work.) □

Problem 21. [16] Find the least positive integer n such that

$$\sum_{m=1}^{n} m^k$$

is a multiple of 11 for all positive integers k.

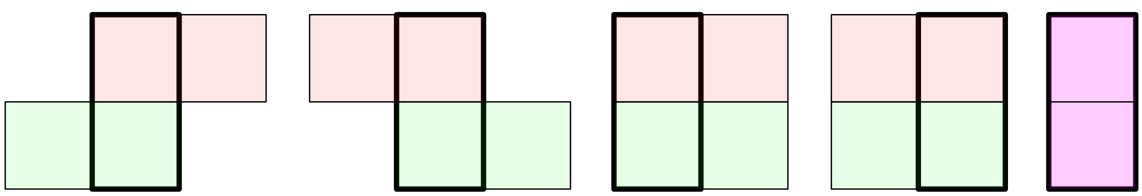
Answer. $\boxed{120}$ *Proposed by Rishabh Das*

Solution. Let $n = 11a + b$, where $0 \le b \le 10$.
Plug in $k = 1$. This means
$$11 \mid \frac{n(n+1)}{2},$$
so $b = 0$ or $b = 10$.
Plug in $k = 10$. Note that $t^{10} \equiv 1 \pmod{11}$ if $11 \nmid t$, and 0 otherwise by Fermat's Little Theorem. Thus, we get
$$\sum_{m=1}^{n} m^{10} \equiv 10a + b \equiv b - a \pmod{11},$$
so $a \equiv b \pmod{11}$.
The two cases are $a \equiv b \equiv 0 \pmod{11}$ and $a \equiv b \equiv 10 \pmod{11}$. The first case gives $n = 121$ and the second case gives $n = 120$. We try 120 first, since it is smaller. (Actually, we can say that if $11 \mid n$ works, then $n - 1$ will also work, so we may discard the first case entirely.)
Suppose $n = 120$. Then
$$\sum_{m=1}^{120} m^k \equiv 11 \left(\sum_{m=1}^{10} m^k \right) \equiv 0 \pmod{11},$$
so $n = 120$ does work. □

Problem 22. **[16]** A 2×14 board is wrapped around to form a cylinder of height 2 and circumference 14. How many ways can you tile the cylinder with 1×2 or 2×1 pieces? Rotations and reflections are considered distinct.

Answer. $\boxed{845}$ *Proposed by Rishabh Das*

Solution. Let the answer for a $2 \times n$ board that isn't on a cylinder be a_n.
Fix one column of the cylinder. We do cases on the ways to fill out this column.



If we have one of the first two cases, then there is one way each, as we must tile the two rows separately. This gives 2 cases.

If we have one of the next two cases, then there are a_{12} ways each, i.e. in total there are $2a_{12}$ ways.

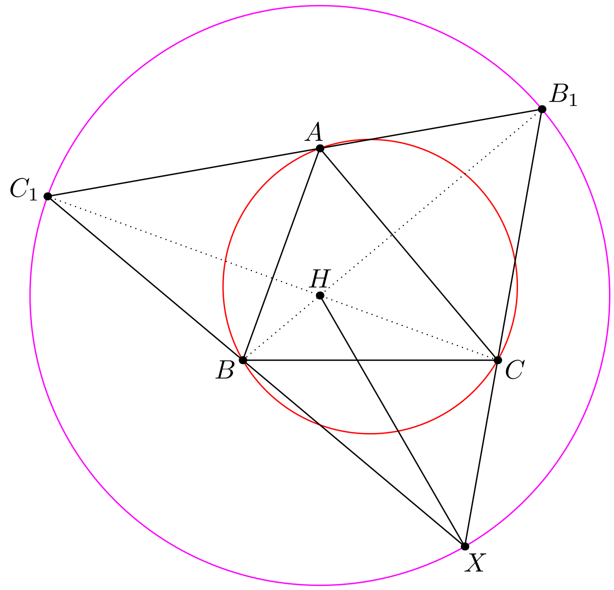
If we have the last case, there are a_{13} ways.

In total, there are $2a_{12} + a_{13} + 2$ ways. We now compute a_n. If we were to do cases on how to fill the first column, then we'd get $a_n = a_{n-1} + a_{n-2}$. Thus, from $a_1 = 1$ and $a_2 = 2$, we see $a_n = F_{n+1}$ where F_k is the kth Fibonacci number. Then the number of ways is $2 \cdot 233 + 377 + 2 = 845$. □

Problem 23. **[17]** In acute $\triangle ABC$, let the orthocenter be H. Let the reflection of B over AC be B_1, and the reflection of C over AB be C_1. Let C_1B intersect B_1C at X. If B_1, A, and C_1 are collinear, $AB = 5$, and $AC = 8$, compute HX.

Answer. $\boxed{\dfrac{13\sqrt{3}}{3}}$ *Proposed by Rishabh Das*

Solution. Since $180^\circ = \angle C_1AB_1 = \angle C_1AB + \angle BAC + \angle CAB_1 = 3\angle A$, so $\angle A = 60^\circ$.



We claim H is the circumcenter of $\triangle B_1C_1X$. We have $\angle C_1B_1X = \angle AB_1C = \angle B$ and similarly $\angle B_1C_1X = \angle C$. Thus, $\triangle XB_1C_1 \sim \triangle ABC$.

However, $\angle C_1B_1H = \angle ABH = 90^\circ - \angle A = 30^\circ$ and similarly $\angle B_1C_1H = 30^\circ$, so $B_1H = C_1H$. Since $\angle B_1HC_1 = 120^\circ = 2\angle B_1XC_1$ and $B_1H = C_1H$, we can conclude H is the circumcenter of $\triangle XB_1C_1$.

Thus, $HX = HB_1 = HC_1$. However, we know $B_1C_1 = 5 + 8 = 13$, so $HB_1 = HC_1 = \frac{13\sqrt{3}}{3}$, as B_1HC_1 is a $30 - 30 - 120$ triangle. Thus, $HX = \frac{13\sqrt{3}}{3}$. □

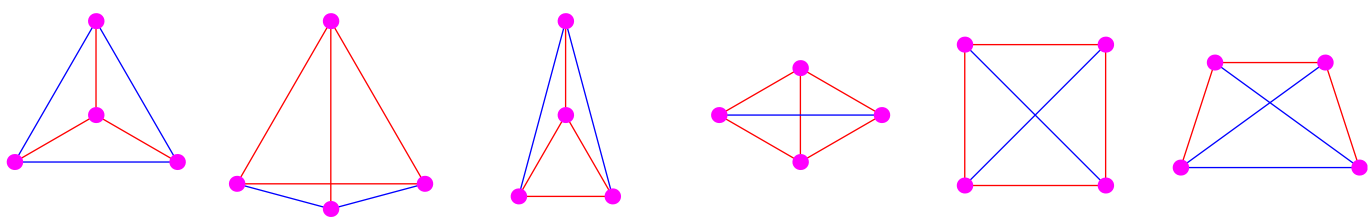
Problem 24. **[17]** Mr. Kats chooses four distinct points A, B, C, and D that lie on a plane. He notices the set

$$\{AB, AC, AD, BC, BD, CD\}$$

is equal to $\{1, x\}$, where $x \neq 1$. How many possible values of x are there?

Answer. $\boxed{8}$ *Proposed by Rishabh Das*

Solution. We first find all configurations that work. We claim the following are all of them:



Label these, from left to right, 1 through 6.

Suppose that one point is a distance r away from all other points. If no pair of the remaining points are a distance of r away from each other, then we get figure 1. If one pair of them are a distance of r away, we get figures 2 and 3. If two pairs of them are a distance of r away, we get figure 4.

Otherwise, suppose that no point is the circumcenter of the remaining three. If 5 or 6 of the distances are the same, this is impossible. If 4 of them are the same, then the only way we don't get a circumcenter is if we get AB, BC, CD, and DA are of the same length. The only way $AC = BD$ then is if it is a square, i.e. figure 5.

Finally, suppose that 3 distances are the same. The only way this can happen without making a circumcenter is if AB, BC, and CD are the same. In order for $AC = BD$, it must be an isosceles trapezoid. Then there is only one way to make $AD = AC = BD$, when the four points are part of a regular pentagon, i.e. figure 6.

Thus, all configurations have been found, and it remains to compute the number of distinct values of x. If the red distance is 1, then the blue distances can be computed as $\sqrt{3}$, $\frac{\sqrt{6}-\sqrt{2}}{2}$, $\frac{\sqrt{6}+\sqrt{2}}{2}$, $\sqrt{3}$, $\sqrt{2}$, $\frac{\sqrt{5}+1}{2}$ respectively. If the blue distance is 1, then the red distance is the reciprocal of these, i.e. $\frac{\sqrt{3}}{3}$, $\frac{\sqrt{6}+\sqrt{2}}{2}$, $\frac{\sqrt{6}-\sqrt{2}}{2}$, $\frac{\sqrt{3}}{3}$, $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{5}-1}{2}$, respectively. (These values don't need to be computed directly, but is done here for completeness. For example, the last configuration can be seen to produce different values than the other figures, without computing any of these values.) Of these, there are 8 distinct values. □

Problem 25. [18] A deck of 2022 cards is riffle-shuffled in the following manner:
The deck is split exactly in half, with the top 1011 cards making up pile 1 and the bottom 1011 cards making up pile 2. The bottom card of pile 2 is placed down, and then the bottom card of pile 1 is placed on top of that. This is then repeated 1010 more times, thus forming a pile of 2022 cards in a different order.
How many times must this process be repeated in order to return the deck to its original order?
(As an example, one shuffle of the 6 cards in the order $1, 2, 3, 4, 5, 6$ would result in the cards being in the order $1, 4, 2, 5, 3, 6$.)

Answer. $\boxed{322}$ *Proposed by Rishabh Das*

Solution. Label the cards $0, 1, 2, \ldots 2021$. Card 2021 never changes position, so we may ignore it. Note that card $1011a + b$, where $0 \le b \le 1010$, will go to position $a + 2b \equiv 2(1011a + b) \pmod{2021}$. Thus, we may think of the position of each card being multiplied by $2 \pmod{2021}$ in each shuffle. Thus, the answer is

$$\operatorname{ord}_{2021} 2 = \operatorname{lcm}(\operatorname{ord}_{43} 2, \operatorname{ord}_{47} 2).$$

We know

$$2^7 = 128 \equiv -1 \pmod{43},$$

so $2^{14} \equiv 1 \pmod{43}$, so $\operatorname{ord}_{43} 2 = 14$.

Now we compute $\operatorname{ord}_{47} 2$. Note that it will divide 46, and it is clearly not 1 or 2, so it is 23 or 46. We can compute

$$2^{23} \equiv 49^{23} \equiv 7^{46} \equiv 1 \pmod{47}$$

by Fermat's Little Theorem, so $\operatorname{ord}_{47} 2 = 23$. The lcm of 14 and 23 is 322. □

Problem 26. [18] Compute

$$\tan\left[\sum_{k=2}^{\infty} \arctan\left(\frac{2k}{2k^4+1}\right)\right].$$

Answer. $\boxed{\frac{1}{5}}$ *Proposed by Rishabh Das*

Solution. We make the algebraic manipulation

$$\frac{2k}{2k^4+1} = \frac{4k}{4k^4+2} = \frac{(2k^2+2k+1)-(2k^2-2k+1)}{1+(2k^2+2k+1)(2k^2-2k+1)}.$$

This means that

$$\arctan\left(\frac{2k}{2k^4+1}\right) = \arctan(2k^2+2k+1) - \arctan(2k^2-2k+1).$$

Let $f(n) = 2n^2 - 2n + 1$. Then

$$f(n+1) = 2(n+1)^2 - 2(n+1) + 1 = (2n^2+4n+2) - (2n+2) + 1 = 2n^2 + 2n + 1.$$

This means that we want to find

$$\tan\left[\sum_{k=2}^{\infty} \arctan(f(k+1)) - \arctan(f(k))\right].$$

Note that

$$\sum_{k=2}^{n} \arctan(f(k+1)) - \arctan(f(k)) = \arctan(f(n+1)) - \arctan(f(2))$$

is a partial sum, and as n goes to infinity $\arctan(f(n+1))$ approaches $\frac{\pi}{2}$. We can also compute $f(2) = 5$, so we want

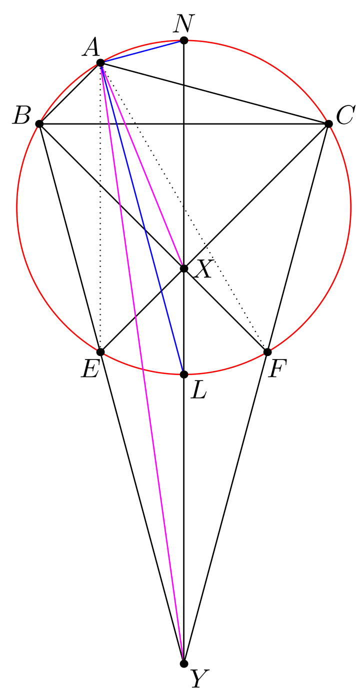
$$\tan\left(\frac{\pi}{2} - \arctan(5)\right) = \frac{1}{5}.$$

□

Problem 27. **[19]** Let the circumcircle of $\triangle ABC$ be Ω. Points E and F are picked on Ω such that $AE \perp BC$ and $\angle AEF = 90^\circ$. Let BF and CE intersect at X, and let BE and CF intersect at Y. If $\frac{AX}{AY} = \frac{1}{2}$, $\angle BAC = 120^\circ$, and $BC = 14$, then compute the area of $\triangle ABC$.

Answer. $\boxed{14\sqrt{3}}$ *Proposed by Rishabh Das*

Solution. Assume $AB < AC$. We claim $\frac{AX}{AY} = \frac{AB}{AC}$.



Let L be the arc midpoint of $\widehat{BC}$ not including A, and N be diametrically opposite L.

Note that $(E, F; L, N) = -1$ since $ELFN$ is a kite. Then we see

$$-1 = (E, F; L, N) \stackrel{C}{=} (X, Y; L, N).$$

However, $\angle NAL = 90^\circ$, so AL must bisect $\angle XAY$. By the angle bisector theorem, $\frac{AX}{AY} = \frac{XL}{LY}$. Since CL bisects $\angle XCY$, we also get $\frac{XL}{LY} = \frac{CX}{CY}$. (We can also note that Ω is an Appolonius circle.) Thus,

$$\frac{AX}{AY} = \frac{CX}{CY}.$$

However, the value of $\frac{CX}{CY}$ can be computed as

$$\frac{\frac{a/2}{\cos(90^\circ - \angle B)}}{\frac{a/2}{\cos(90^\circ - \angle C)}} = \frac{\sin\angle C}{\sin\angle B} = \frac{AB}{AC},$$

so the claim is proven.

Thus, $\frac{AB}{AC} = \frac{AX}{AY} = \frac{1}{2}$, so $AB = x$ and $AC = 2x$ for some x. By Law of Cosines on $\triangle ABC$, we see

$$x^2 + (2x)^2 + x(2x) = 7x^2 = 196 \implies x^2 = 28.$$

The area is

$$\frac{1}{2} \cdot x \cdot 2x \cdot \frac{\sqrt{3}}{2} = \frac{x^2\sqrt{3}}{2} = 14\sqrt{3}.$$

□

Problem 28. [19] There are infinitely many boxes in a line, labeled with the integers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots,$$

in that order. A ball is placed in the boxes whose label has absolute value at most 2021. Every second, one of the 4043 balls is selected at random, and a coin is flipped. If the coin lands heads, the selected ball moves one box to the right, and if it lands tails it moves one box to the left. After 2021 seconds, the expected number of balls in the box labeled "2" is $1 - \frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

Answer. $\boxed{16337765}$ *Proposed by Rishabh Das*

Solution. The probability that a ball moves from box k to box 2 is the same as the probability that a ball from box 2 moves to box k. Thus, we want the probability that the ball from box 2 moves to a box in the range $[-2021, 2021]$. The only way it ends up outside of this range is if ends up at 2022 or 2023. In order to end up at 2023, it must move right every single move, which happens with probability $\frac{1}{8086^{2021}}$. In order to end up at 2022, it must move right every single move, besides 1 move where some other ball moves. There are $4042 \cdot 2 \cdot 2021 = 4042^2$ ways to have this happen: $4042 \cdot 2$ ways to choose the other move, and 2021 ways to choose when this move happens. Thus, the final probability is

$$1 - \frac{m}{n} = 1 - \frac{4042^2 + 1}{8086^{2021}} \implies \frac{m}{n} = \frac{4042^2 + 1}{8086^{2021}}.$$

The numerator is odd, and 4043 also clearly does not divide the numerator. Thus, $m = 4042^2 + 1 = 16337765$. □

Problem 29. [20] Consider the polynomial $x^{12} - 1$ with coefficients in $\mathbb{F}_{31}$. Let there be N polynomials $P(x)$ in $\mathbb{F}_{31}$ with degree less than 12 such that $P(x)$ and $x^{12} - 1$ have no common non-constant factor in $\mathbb{F}_{31}[x]$. Compute the sum of the (not necessarily distinct) prime factors of N.
(For example, if $N = 12 = 2^2 \cdot 3$, you should submit $2 + 2 + 3 = 7$.)

Answer. $\boxed{120}$ *Proposed by Srinath Mahankali*

Solution. We first fully factor $x^{12} - 1$ in $\mathbb{F}_{31}$. We can immediately get a difference of squares as $(x^6 - 1)(x^6 + 1)$. Let ω be a primitive 6th root, which exists since $31 \equiv 1 \pmod{6}$. Then $x^6 - 1$ has roots $1, \omega, \omega^2, \omega^3, \omega^4, \omega^5$. Let z be a value such that $z^3 \equiv -1 \pmod{31}$, which exists since we can take, for example, g^5 where g is a primitive root. Then $x^6 + 1$ factors as $(x^2 + 1)(x^2 + z)(x^2 + z^2)$. None of these quadratic factors can factor further, as this would imply that

$$y^6 \equiv -1 \pmod{31} \implies y^{30} \equiv -1 \pmod{31},$$

impossible by Fermat's Little Theorem. Thus,

$$x^{12} - 1 = (x - 1)(x - \omega)(x - \omega^2)(x - \omega^3)(x - \omega^4)(x - \omega^5)(x^2 + 1)(x^2 + z)(x^2 + z^2).$$

We just need P to be relatively prime to all these polynomials. By CRT, we can consider each polynomial separately.

There are $31^1 - 1 = 30$ possible residues mod a linear polynomial, and $31^2 - 1 = 960$ possible residues mod a quadratic polynomial. Thus, there are

$$30^6 \cdot 960^3 = 2^{24} 3^9 5^9$$

possible polynomials P, which makes the answer $24 \cdot 2 + 9 \cdot 3 + 9 \cdot 5 = 120$. □

Problem 30. [**20**] For all positive integers n, let $P_n(x)$ denote the unique polynomial with degree at most n such that

$$P_n(m) = \binom{n}{m} \text{ for all integers } 0 \le m \le n.$$

Let a_n denote the leading coefficient of P_n. Compute $\dfrac{a_{2021}}{a_{2022}}$.

Answer. $\boxed{-1022121}$ *Proposed by Rishabh Das*

Solution. We use the fact that the nth order finite difference is $n!$ times the coefficient of x^n in a polynomial of degree n.

The nth order finite difference is

$$\binom{n}{0}\binom{n}{n} - \binom{n}{1}\binom{n}{n-1} + \cdots + (-1)^n\binom{n}{n}\binom{n}{0}.$$

Note that if we expand $(1-x)^n(1+x)^n$, this is the coefficient of x^n. Thus, this is equal to the x^n coefficient in $(1-x^2)^n$. If n is odd then this is 0, while if n is even this is

$$\binom{n}{n/2}(-1)^{n/2}.$$

This divided by $n!$ gives

$$a_n = \frac{(-1)^{n/2}}{(n/2)!^2}$$

A leading coefficient of 0 is not allowed, so when n is odd we look at the $(n-1)$sth order finite difference. This is

$$\binom{n-1}{0}\binom{n}{n} - \binom{n-1}{1}\binom{n}{n-1} + \cdots + (-1)^{n-1}\binom{n-1}{0}\binom{n}{1}.$$

If we were to expand $(1-x)^{n-1}(1+x)^n$ then this would be the coefficient of x^n. Thus, this is equal to the coefficient in x^n in $(1-x^2)^{n-1}(1+x)$. Since n is odd, this is equal to the coefficient of x^{n-1} in $(1-x^2)^{n-1}$, which is

$$\binom{n}{(n-1)/2}(-1)^{(n-1)/2}.$$

This divided by $(n-1)!$ gives

$$a_n = \frac{(-1)^{(n-1)/2}}{((n-1)/2)!^2}.$$

The final answer is

$$\frac{(-1)^{1010}}{1010!^2} \cdot \frac{1011!^2}{(-1)^{1011}} = -1011^2 = -1022121.$$

□