Stuyvesant Team Contest: Problems

Winter 2021

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1. **[6]** A point is chosen inside a unit square at random. What is the probability that the distance from this point to the closest side is at most $\frac{1}{2}$?

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2. **[6]** Two positive integers x and y sum up to 41, with $x < y$. Find the maximum possible value of $100x + y$.

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3. **[7]** Triangle ABC has $AB = AC = 29$ and perimeter 98. What is the area of the triangle?

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4. **[7]** Mr. Kats writes the letters

STCSTC

on a board. Mr. Sterr sees this, and erases three of the letters. How many ways can he do this so that the three remaining letters read "STC" in that order?

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5. **[8]** Equilateral triangle $\triangle ABC$ has side length 2. A point P is selected on segment BC such that $\frac{BP}{PC} = \frac{20}{21}$. Points X and Y are selected on AB and AC respectively such that $PX \perp AB$ and $PY \perp AC$. Compute $AX + AY$.

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6. **[8]** What's the largest positive integer that can't be written as the sum of two composite positive integers?

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7. **[9]** The sequence a, b, c is an increasing arithmetic sequence of positive integers and $a, b, c+1$ is an increasing geometric sequence of positive integers. Given that b is at least 2021, find the smallest possible value of b.

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8. **[9]** Compute

$$\frac{(1 + 2020^2 + 2021^2)^2}{2020^2 + 2021^2 + (2020 \cdot 2021)^2}.$$

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9. **[10]** The numbers $15, 25, 35, 45, 55, 65, 75, 85$, and 95 are written on pieces of paper in a hat. Mr. Cocoros grabs two distinct pieces of paper from the hat, and computes the product of the numbers. What is the probability that the tens digit of the result is 2?

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10. **[Up to 10]** There are N teams competing in STC. Pick an integer X between 1 and 10, inclusive. Let k teams (including your team) pick X. If $k > \frac{N}{10}$, you will receive 0 points. Otherwise, you will receive X points.

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11. **[11]** Let $S = \{1, 2, ..., 20\}$. Mr. Cocoros chooses a two-element subset of S at random. What is the expected value of the positive difference between the two elements?

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12. **[11]** Let AB be the diameter of a semicircle, ω, with $AB = 4$. Let O be the midpoint of AB, and C be the midpoint of OB. The perpendicular to AB through C intersects ω again at D. A circle is drawn tangent to segments CD and CB, and to arc BD of ω. What is the radius of this circle?

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13. **[12]** A subset S of $\{1, 3, 5, \ldots 2021\}$ is chosen such that if a and b are any distinct elements of S, then a does not divide b. What is the maximum possible size of S?

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14. **[12]** The equation $x^2 - 2x + c = 0$ has roots $\tan\theta$ and $\sec\theta$ for some $\theta \in (0°, 90°)$. Compute c.

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15. **[13]** For $n \geq 2$, define $f(n)$ as the largest proper divisor d of n for which the number of positive integer divisors of d is maximized. For which k is

$$\underbrace{f(f(f(\cdots f(}_{k\ fs}10!)\cdots))) = 1,$$

where f is iterated k times?

Note: A *proper divisor* of n is a divisor of n that is not n.

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16. **[13]** A square is centered at the origin. Every second, Mr. Sterr selects a vertex of the square at random, and the square is reflected over the selected vertex. What is the probability that after 4 seconds, the square is in its original position?

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17. **[14]** A sphere goes through the point $O = (0, 0, 0)$ and is tangent to the plane $x + y + z = 1$ at the point $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Suppose this sphere intersects the x, y, and z axes at the points X, Y, and Z respectively, where $O \neq X, Y, Z$. Compute the volume of tetrahedron $OXYZ$.

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18. **[14]** If $x > 1$ and

$$\log_5(\log_5 x) + 4\log_x(\log_x 5) = 0,$$

find the sum of all possible values of x.

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19. **[15]** Call a rectangle *n-special* if it has integer dimensions, and the sum of its area and perimeter is equal to n. Compute the sum of the smallest two values of n for which there are exactly 5 noncongruent n-special rectangles.

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20. **[Up to 64]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive 2^n points for n correct answers, but you will receive 0 if any of the questions you choose to answer is answered incorrectly. Note that this means if you submit "XXXXXX" you will get one point.

(1) A *perfect power* is a number of the form n^k for an integer $k \geq 2$ and a positive integer n. There are infinitely many perfect powers that are also palindromes.

(2) If $\tau(n)$ is equal to the number of positive divisors of n, then

$$\sum_{k=1}^{\infty} \frac{(\tau(k))^{2021}}{k^2}$$

diverges.

(3) There exists a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

$$f(x) + f(y) \leq x - y$$

for all $x, y \in \mathbb{R}$.

(4) There exists a set $\mathcal{S}$ of squares, each with finite, positive area in $\mathbb{R}^2$ such that every point in $\mathbb{R}^2$ is part of exactly one square in $\mathcal{S}$. (Here a square does not consist of its interior, but only its border.)

(5) A particle on the number line starts at 0. Every second, it either moves forward by 1 or backwards by 1 at random. Let p_n be the probability that after n seconds, the absolute value of the position of the particle is at most $n^{2020/2021}$. Then

$$\lim_{n \longrightarrow \infty} p_n = 0.$$

(6) An $n \times n$ square is tiled with L-shaped trinominoes. Then 6 must divide n. (An L-shaped trinomino is a 2×2 square with one of the 1×1 corners cut out.)

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21. **[16]** Find the least positive integer n such that

$$\sum_{m=1}^{n} m^k$$

is a multiple of 11 for all positive integers k.

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22. **[16]** A 2×14 board is wrapped around to form a cylinder of height 2 and circumference 14. How many ways can you tile the cylinder with 1×2 or 2×1 pieces? Rotations and reflections are considered **distinct.**

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23. **[17]** In acute $\triangle ABC$, let the orthocenter be H. Let the reflection of B over AC be B_1, and the reflection of C over AB be C_1. Let C_1B intersect B_1C at X. If B_1, A, and C_1 are collinear, $AB = 5$, and $AC = 8$, compute HX.

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24. **[17]** Mr. Kats chooses four distinct points A, B, C, and D that lie on a plane. He notices the set

$$\{AB, AC, AD, BC, BD, CD\}$$

is equal to $\{1, x\}$, where $x \neq 1$. How many possible values of x are there?

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25. **[18]** A deck of 2022 cards is riffle-shuffled in the following manner:

The deck is split exactly in half, with the top 1011 cards making up pile 1 and the bottom 1011 cards making up pile 2. The bottom card of pile 2 is placed down, and then the bottom card of pile 1 is placed on top of that. This is then repeated 1010 more times, thus forming a pile of 2022 cards in a different order.

How many times must this process be repeated in order to return the deck to its original order?

(As an example, one shuffle of the 6 cards in the order $1, 2, 3, 4, 5, 6$ would result in the cards being in the order $1, 4, 2, 5, 3, 6$.)

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26. **[18]** Compute

$$\tan\left[\sum_{k=2}^{\infty} \arctan\left(\frac{2k}{2k^4+1}\right)\right].$$

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27. **[19]** Let the circumcircle of $\triangle ABC$ be Ω. Points E and F are picked on Ω such that $AE \perp BC$ and $\angle AEF = 90°$. Let BF and CE intersect at X, and let BE and CF intersect at Y. If $\frac{AX}{AY} = \frac{1}{2}$, $\angle BAC = 120°$, and $BC = 14$, then compute the area of $\triangle ABC$.

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28. **[19]** There are infinitely many boxes in a line, labeled with the integers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots,$$

in that order. A ball is placed in the boxes whose label has absolute value at most 2021. Every second, one of the 4043 balls is selected at random, and a coin is flipped. If the coin lands heads, the selected ball moves one box to the right, and if it lands tails it moves one box to the left. After 2021 seconds, the expected number of balls in the box labeled "2" is $1 - \frac{m}{n}$, where m and n are relatively prime positive integers. Compute m.

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29. **[20]** Consider the polynomial $x^{12} - 1$ with coefficients in $\mathbb{F}_{31}$. Let there be N polynomials $P(x)$ in $\mathbb{F}_{31}$ with degree less than 12 such that $P(x)$ and $x^{12} - 1$ have no common non-constant factor in $\mathbb{F}_{31}[x]$. Compute the sum of the (not necessarily distinct) prime factors of N.

(For example, if $N = 12 = 2^2 \cdot 3$, you should submit $2 + 2 + 3 = 7$.)

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30. **[20]** For all positive integers n, let $P_n(x)$ denote the unique polynomial with degree at most n such that

$$P_n(m) = \binom{n}{m} \text{ for all integers } 0 \leq m \leq n.$$

Let a_n denote the leading coefficient of P_n. Compute $\frac{a_{2021}}{a_{2022}}$.

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