Solutions to the Stuyvesant Team Contest

Fall 2019

Problem 1. **[6]** Compute the smallest positive integer n such that if n students participate in the Stuyvesant Team Contest, they can be split evenly into $1, 2, 3, 4, 5, 6,$ and 7 teams.

Answer. $\boxed{420}$ *Proposed by Kimi*

Proof. We seek the least common multiple of $1, 2, 3, 4, 5, 6, 7$, which is 420. □

Problem 2. **[6]** $ABCD$ is a trapezoid has area 276 and $AB \parallel CD$. Points M and N are midpoints of segment AD and BC, respectively. Compute the area of quadrilateral $DMBN$.

Answer. $\boxed{138}$ *Proposed by Kimi*

Proof. Connect segment BD. The midpoint conditions give $[BMD] = \frac{1}{2}[ABD]$ and $[BDN] = \frac{1}{2}[BDC]$. Summing gives $[DMBN] = \frac{1}{2}[ABCD] = 138$. □

Remark. The condition $AB \parallel CD$ is not necessary.

Problem 3. **[7]** Given that real number r satisfies $|2019 - r| + \sqrt{r - 2020} = r$, compute all possible values of $r - 2019^2$.

Answer. $\boxed{2020}$ *Proposed by Kimi*

Proof. Since $r - 2020$ is under the square root, $r \geq 2020 > 2019$. Solving $r - 2019 + \sqrt{r - 2020} = r$ gives $r = 2020 + 2019^2$, so the answer is 2020. □

Problem 4. **[7]** How many distinct 8 digit numbers can be formed by concatenating exactly one of each of $\{2, 0, 1, 9, 2019\}$?

Answer. $\boxed{95}$ *Proposed by Kimi*

Proof. Since 0 cannot be the leading digit, the total number of ways to order the 5 numbers is $4 \times 4 \times 3 \times 2 \times 1 = 96$. Note, we counted 20192019 twice, so the answer is $96 - 1 = 95$. □

Problem 5. **[8]** If x and y are positive reals such that:

$$x + y^2 = 2019$$
$$x^2 + y^2 = 2109$$

then compute $x^3 + y^2$

Answer. $\boxed{3009}$ *Proposed by Rishabh*

Proof. Subtracting the first equation from the second gives $x^2 - x = 90$. This means $x = -9$ or $x = 10$. Since $x > 0$, $x = 10$. This means $y^2 = 2009$. The desired sum is 3009. □

Problem 6. **[8]** Let $f(x) = \frac{x}{\sqrt{1+x^2}}$ and $f_n(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots))$, compute $f_{99}(1)$.

Answer. $\boxed{\dfrac{1}{10}}$ *Proposed by Kimi*

Proof. Notice the pattern $f_n(1) = \frac{1}{\sqrt{n+1}}$. We prove this via induction: the base case $f_1(1) = f(1) = \frac{1}{\sqrt{2}}$ holds. Assuming the case of $f_n(1)$, we have

$$f_{n+1}(1) = f(f_n(1)) = f\left(\frac{1}{\sqrt{n+1}}\right) = \frac{\frac{1}{\sqrt{n+1}}}{\sqrt{1+\frac{1}{n+1}}} = \frac{1}{\sqrt{n+2}}$$

as desired. Thus, $f_n(1) = \frac{1}{\sqrt{n+1}}$ so $f_{99}(1) = \frac{1}{\sqrt{99+1}} = \frac{1}{10}$. □

Problem 7. **[9]** Equilateral $\triangle ABC$ has side-length 2019. Points X, Y, and Z are on segments BC, CA, and AB, respectively. If $CX = CY = BZ = 673$, compute the radius of the circle passing through X, Y, and Z.

Answer. $\boxed{673}$ *Proposed by Akash*

Proof. Since $BX = 2019 - 673 = 1346 = 2BZ$, we see that triangle BXZ is a $30-60-90$ triangle, so $\angle BXZ = 30°$. Additionally, since $CX = CY$, triangle CXY is equilateral, so $\angle CXY = 60°$. Thus, $\angle YXZ = 180° - 30° - 60° = 90°$. Thus, ZY is a diameter of the circle passing through X, Y, and Z. Observe that $ZY = AZ = AB - BZ = 2019 - 673 = 1346$. Thus, the radius is $\frac{1}{2} \cdot 1346 = 673$ □

Problem 8. **[9]** A soccer ball is glued together edge-to-edge from 32 shapes, each of which either a pentagon or a hexagon. Given each pentagon is glued to 5 hexagons and each hexagon is glued to 3 pentagons and 3 hexagons, compute the number of hexagons.

Answer. $\boxed{20}$ *Proposed by Kimi*

Proof. Let n be the number of hexagons, then there are $32-n$ pentagons. The number of pentagon-hexagon-glued edges is $3n = 5(32-n)$. Thus, $n = 20$. □

Problem 9. **[10]** Compute the sum of all possible non-negative integers n such that $0!+1!+\cdots+n!$ is a perfect square.

Answer. $\boxed{2}$ *Proposed by Akash*

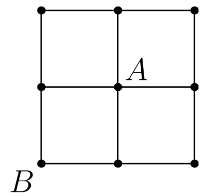
Proof. First, verify $n = 0$ and $n = 2$ work, but $n = 1$ and $n = 3$ do not. Now, for $n \geq 4$, since $k! \equiv 0 \pmod{4}$ for $k \geq 4$, $\sum_{k=0}^{n} k! \equiv 1+1+2+6 \equiv 2 \pmod{4}$, so it cannot be a perfect square. □

Problem 10. **[10]** The number of teams is N. Submit an integer a between 0 and N, inclusive. Let A be the average of all submissions and n be the number of submissions greater than A. You will receive $\left\lceil \frac{20}{2+|a-n|} \right\rceil$ points.

Answer. $\boxed{\text{N.A.}}$ *Proposed by Ethan*

Remark. During the contest, $N = 22$, $A = TBA$, and $n = TBA$.

Problem 11. [11] An ant starts at point A. Every second, picks a point that it is adjacent to at random, and moves to this point. (Adjacent means connected by an edge.) What is the probability that after 2020 seconds, the ant is on point B?



Answer. $\boxed{\frac{1}{6}}$ *Proposed by Akash*

Proof. Note a simple parity observation: after 2019 (any odd number) seconds, it is at any one of the 4 the midpoints of the sides, i.e. points adjacent to A. Then, the next second, it can move to either of 2 corners or A, so the probability of the ant at A after 2020 seconds is $\frac{1}{3}$. Since the four corners are symmetric, the probability of the ant at B after 2020 seconds is simply $(1-\frac{1}{3})/4 = \frac{1}{6}$. □

Problem 12. [11] Let $AB = 20, BC = 29$, and $CA = 21$. Reflect A over BC to get A'. Reflect A' over AB and AC to get X and Y, respectively. Find the area of quadrilateral $XYCB$.

Answer. $\boxed{630}$ *Proposed by Akash*

Proof. Let AA' intersect BC at point D. Let the area of ADB equal M and the area of ACD equal N. Note that $\angle A'AX = 2\angle A'AB$ and $\angle A'AY = 2\angle A'AC$. Adding these two together, we get $\angle A'AX + \angle A'AY = 2\angle A'AB + 2\angle A'AC = 2(\angle A'AB + \angle A'AC) = 2(\angle BAC) = 180°$. Thus, X, A, Y are collinear. Therefore, $[YCBX] = [YCA] + [ACB] + [ABX]$. Note that $[ABX] = [ABA'] = 2[ABD] = 2M$, and similarly, $[ACY] = [ACA'] = 2[ACD] = 2N$. Thus, $[YCBX] = [YCA] + [ACB] + [ABX] = 2M + [ACB] + 2N$. Note that $M + N = [ABD] + [ACD] = [ACB]$, so the answer is simply $3[ACB] = 3 \cdot \frac{20\cdot 21}{2} = 630$ □

Problem 13. [12] Compute the number of pairs of positive integers (m, n) such that $m, n \leq 30$

$$m + \gcd(m, n) = n + \operatorname{lcm}(m, n)$$

Answer. $\boxed{111}$ *Proposed by Akash*

Proof. Let $gcd(m, n) = d$, and let $m = da$ and $n = db$ for positive integers a and b. Note that we have $\gcd(a, b) = 1$. Additionally, note that $\operatorname{lcm}(m, n) = d\operatorname{lcm}(a, b) = dab$ Our equation now becomes $da + d = db + dab$. Dividing by d and rearranging gives $ab + b - a - 1 = 0$. Factoring the left side gives us $(a + 1)(b - 1) = 0$. Since $a > 0$, we get $b = 1$. Thus, our condition is true if and only if m is a multiple of n. There are $\lfloor \frac{30}{k} \rfloor$ ordered pairs (m, n) such that $m = kn$. Summing this over all possible values of k gives us

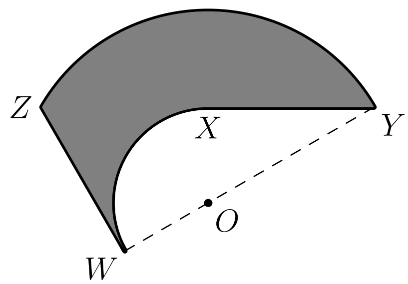
$$\sum_{k=1}^{30} \left\lfloor \frac{30}{k} \right\rfloor = 30 + 15 + 10 + 7 + 6 + 5 + 4 + 3 + 3 + 3 + 5 \cdot 2 + 15 \cdot 1 = 111$$ □

Problem 14. [12] If a is selected from $\{1, 2, \ldots, 10\}$ uniformly randomly and b is independently (of a) selected from $\{-10, -9, \ldots, -1\}$ uniformly randomly, compute the probability $a^2 + b$ is a multiple of 3.

Answer. $\boxed{\frac{37}{100}}$ *Proposed by Kimi*

Proof. Note, a^2 is congruent to 0 or 1 (mod 3). If $a \equiv 0 \pmod 3$, b must also be $0 \pmod 3$. There are 3 choices of a and 3 choices of b; if $a \equiv \pm 1 \pmod 3$, b must be $2 \pmod 3$. There are 7 choices of a and 4 choices of b. Thus, the desired probability is $\frac{3\times 3+4\times 7}{10\times 10} = \frac{37}{100}$. □

Problem 15. **[13]** In the diagram below, arcs WX and YZ both have center O with radii 1 and 2. Given $\angle OXY = \angle OWZ = 90^\circ$ and points W, O, Y are collinear, compute the area of the shaded region $WXYZ$.



Answer. $\boxed{\pi}$ *Proposed by Kimi*

Proof. Connect segment OX. Note $OY = 2 = 2\times OX$, $\triangle OXY$ is a 30-60-90 triangle so $\angle XOW = 120^\circ$. Rotate the figure about O clockwise 120° and 240°. We see three of the shaded figures cover the ring between circles of radii 1 and 2 centered at O exactly. Thus, the answer is $\frac{\pi(2^2-1^2)}{3} = \pi$. □

Remark. Look at the Google Chrome icon.

Problem 16. **[13]** The roots of $x^3 - 14x^2 + 54x - p = 0$ are positive real numbers that form a right triangle. Find p.

Answer. $\boxed{196\sqrt{11} - 616}$ *Proposed by Akash*

Proof. Let the roots be a, b, c, where c would be the hypotenuse of the right triangle. Then $a^2 + b^2 + c^2 = 2c^2$. However, we know that

$$a^2 + b^2 + c^2 = (a+b+c)^2 - 2(ab+bc+ca) = 196 - 108 = 88$$

by Vieta's, so $2c^2 = 88$. Thus, we know $c = 2\sqrt{11}$. Plugging in $2\sqrt{11}$ into our cubic, we see

$$88\sqrt{11} - 14\cdot 44 + 108\sqrt{11} - p = 0 \implies p = 196\sqrt{11} - 616 \quad \square$$

Problem 17. **[14]** Point D is on side BC of $\triangle ABC$ such that $AD \perp BC$, $AB + BD = DC$, and $\angle B = 40^\circ$. Compute $\angle C$.

Answer. $\boxed{20}$ *Proposed by Kimi*

Proof. Reflect point B across D to obtain point B' and connect AB'. Note, $\triangle ABB'$ is isosceles so $\angle AB'B = \angle B = 40^\circ$ and $AB = AB'$. Then, $CB' = CD - DB' = AB + BD - DB' = AB = AB'$ so $\triangle AB'C$ is also isosceles. Thus, $\angle C = \frac{1}{2}\angle AB'B = 20^\circ$. □

Problem 18. **[14]** Find the last two digits of the sum of all positive integers x such that $(\sqrt{x} + \sqrt{x+2019})^2$ is an integer.

Answer. $\boxed{67}$ *Proposed by Rishabh*

Proof. Expanding the given means that $\sqrt{x(x+2019)}$ is an integer. Let $\gcd(x, x+2019) = \gcd(x, 2019) = d$, and let $x = da, x+2019 = db$. Then we see $d\sqrt{ab}$ is an integer, so $\sqrt{ab}$ is an integer. However, note that a and be are relatively prime. Since their product is a square, both a and b must be squares. Thus, let $x = da_1^2$ and $x + 2019 = db_1^2$. We will now do cases on d, as we know that d is a divisor of 2019.

If $d = 1$ then $b_1^2 - a_1^2 = 2019$, so $(b_1 - a_1)(b_1 + a_1) = 2019$. Then, since $2019 = 1 \cdot 2019 = 3 \cdot 673$, we see that $(b_1, a_1) = (1010, 1009), (338, 335)$, so $x = a_1^2 = 1009^2, 335^2$.

If $d = 3$ then $b_1^2 - a_1^2 = 673$, so $(b_1 - a_1)(b_1 + a_1) = 673$. Then, since $673 = 1 \cdot 673$, we see that $(b_1, a_1) = (337, 336)$, so $x = 3a_1^2 = 3 \cdot 336^2$.

If $d = 673$ then $b_1^2 - a_1^2 = 3$, so $a_1 = 1$ and $b_1 = 2$. This means $x = 673 \cdot 1^2 = 673$.

If $d = 2019$ then $b_1^2 - a_1^2 = 1$, which has no positive integer solutions.

The sum of all numbers is $1009^2 + 335^2 + 3 \cdot 336^2 + 673 \equiv 81 + 25 + 3 \cdot 96 + 73 \equiv 67 \pmod{100}$. □

Problem 19. [15] Real numbers x and y satisfy $-\frac{\pi}{2} < y < 0 < x < \frac{\pi}{2}$. Given $\sin y + \tan^2 x = \sin x + \tan^2 y$ and $\sin^2 x + 2\cos(x-y) + \sin^2 y = \frac{17}{16}$, compute $\sin x + \sin y$.

Answer. $\boxed{\frac{1}{4}}$ *Proposed by Akash*

Proof. Using the condition, we apply some elementary trigonometry to obtain:

$$\sin x - \sin y = \tan^2 x - \tan^2 = \frac{\sin^2 x}{1 - \sin^2 x} - \frac{\sin^2 y}{1 - \sin^2 y} = \frac{\sin^2 x - \sin^2 y}{(1 - \sin^2 x)(1 - \sin^2 y)}$$

so $\sin x + \sin y = (\cos^2 x)(\cos^2 y)$, squaring gives $\sin^2 x + 2\sin x \sin y + \sin^2 y = \cos^4 x \cos^4 y$. Then,

$$\frac{17}{16} = \sin^2 x + 2\cos(x-y) + \sin^2 y = \cos^4 x \cos^4 y + 2\cos x \cos y$$

Solving gives $\cos x \cos y = \frac{1}{2}$ and $\sin x + \sin y = \frac{1}{4}$. □

Problem 20. [Up to 64] Welcome to **USAYNO**!

(1) Define a magic square to be a 3-by-3 square of distinct numbers such that all rows, columns, and the two diagonals diagonal have the same sum. Define a unit fraction as a fraction of the form $\frac{1}{n}$ for a positive integer n. There exists a magic square consisting of only unit fractions.

(2) A knight's tour is a sequence of moves of a knight on a chessboard such that the knight visits every square only once. If the knight ends on a square that is one knight's move from the beginning square (so that it could tour the board again immediately, following the same path), the tour is closed. There exists a closed knight's tour on a 4×4 chessboard.

(3) There exists a closed two-dimensional shape with three non-concurrent lines of symmetry.

(4) Call a function $f : \mathbb{R} \to \mathbb{R}$ *goofy* if $|f(x) - f(y)|^{2019} \le |x - y|^{2020}$ for all reals x and y. Then, every *goofy* function must be a constant function.

(5) Let $\mathcal{S}$ be the set of positive integers with no two consecutive digits that sum up to 9. Then,

$$\sum_{x \in \mathcal{S}} \frac{1}{x}$$

diverges, i.e. for every M, there exist finite subset $\mathcal{T} = \{t_1, t_2, \ldots, t_n\} \subset \mathcal{S}$ such that $\sum_{k=1}^{n} \frac{1}{t_k} \ge M$.

(6) In regular tetrahedron $ABCD$, if points X and Y are chosen on faces ABC and BCD, then there must exists a triangle with side lengths AY, DX, and XY.

Answer. $\boxed{\text{TFFTFT}}$ *Proposed by Akash*

Proof. (1): take the 3-by-3 magic square. Divide each entry by 9! so $k \mapsto \frac{k}{9!}$ is a unit fraction.

(2): Assume that such a tour is possible. Without loss of generality, let it start in the upper left corner. Label the top row as Row 1, the next row as Row 2, and third from top row as Row 3, and the bottom row as Row 4. Note that a knight on Row 1 or Row 4 must go to either Row 2 or Row 3. However, since there are an equal number of squares on the top and bottom row as there are in the middle two rows, we know that the tour must alternate between Row 1/4 and Row 2/3. Thus, there must be an even number of moves taken to get from the starting square to the square immediately to the right of it. However, if we color this chessboard with a checkerboard pattern, these two squares are opposite colors, but knights switch colors every move, so a knight would need to take an odd number of moves. Thus, we have a contradiction.

(3): the center of mass exists by compactness and lies on all lines of symmetry.

(4): we show that if f satisfies $|f(x) - f(y)| \leq C|x-y|^\alpha \ \forall\ x, y \in \mathbb{R}$ for constant $C \in \mathbb{R}$ and $\alpha > 1$ (i.e. f is Hölder continuous of order $\alpha > 1$), it must be constant. Fix x, since $\alpha - 1 > 0$:

$$\Big|\frac{f(y)-f(x)}{y-x}\Big| < C|x-y|^{1-\alpha} \xrightarrow[y\to x]{} 0 \implies \lim_{y\to x}\frac{f(y)-f(x)}{y-x} = 0$$

Thus, f is differentiable at every $x \in \mathbb{R}$ and $f'(x) = 0$, so f is constant. The conclusion follows.

(5): Consider all the d-digit numbers in $\mathcal{S}$. We can pick the first digit in 9 ways. The next digit has 9 choices, and then so does the next digit, and so on. Thus, there are 9^d d-digit numbers in $\mathcal{S}$. Note that if x is a d-digit number in $\mathcal{S}$, then $\frac{1}{x} \leq \frac{1}{10^{d-1}}$. Thus:

$$\sum_{x\in\mathcal{S}}\frac{1}{x} \leq \sum_{d=1}^{\infty}\frac{9^d}{10^{d-1}} = 10\sum_{d=1}^{\infty}\left(\frac{9}{10}\right)^d$$

The latter is a geometric sequence, which converges. Thus, the original sum must converge as well.

(6): Pick a point E in the fourth dimension that is equidistant from points A, B, C, D. Then $ABCD, ABCE, ABDE, ACDE, BCDE$ are all congruent regular tetrahedrons. Thus, $DX = EX$ and $AY = EY$. Then we are asked if there must exist a triangle with side lengths EY, EX, and XY. However, $\triangle EXY$ is such a triangle. □

Problem 21. [16] For any three digit number, define its *Jerry's* to be the set of all distinct two digit integers formed with exactly one copy of each of its digits. Let $\mathcal{J}$ be all three digit numbers equal to the sum of its Jerry's. Compute the product of the mean of $\mathcal{J}$ and the median of $\mathcal{J}$.

Answer. $\boxed{69696}$ *Proposed by Kimi*

Proof. If all three digits are the same, i.e. $N = \overline{XXX}$, its only Jerry is $\overline{XX}$, which is too small. If two of the digits of N are the same, i.e. its digits are X, X, Y, we see the sum of its Jerry's is $\overline{XX} + \overline{XY} + \overline{YX} = 22X + 11Y$, a multiple of 11. Using the divisibility rule of 11, either $N = \overline{XXY}$ and $Y = 0$, in which case $110X = 22X$ is impossible, or $N = \overline{XYX}$ and $Y \equiv 2X \pmod{11}$. Since $0 \leq X, Y \leq 9$, $Y = 2X$ so $22X + 11(2X) = 101X + 10(2X)$ gives no solutions.

Thus, all 3 digits of $N = \overline{XYZ}$ are distinct. Now, we claim all its digits are nonzero: if $Y = 0$, the Jerry's of $N = \overline{X0Z}$ are $\{\overline{X0}, \overline{Z0}, \overline{XZ}, \overline{ZX}\}$, so $100X + Z = 21X + 21Z \implies 79X = 20Z$ is absurd as the leading digit $X \neq 0$. If $Z = 0$, we have $100X + 10Y = 21X + 21Y \implies 79X = 11Z$ is also absurd. Thus, $X, Y, Z \neq 0$, so the Jerry's of N is $\{\overline{XY}, \overline{XZ}, \overline{YX}, \overline{YZ}, \overline{ZX}, \overline{ZY}\}$. Therefore,

$$100X + 10Y + Z = 22X + 22Y + 22Z \implies 26X = 4Y + 7Z \leq 4\times 8 + 7\times 9 = 95 \implies X \leq 3$$

Cases $X = 1, 2, 3$ give $\mathcal{J} = \{132, 264, 396\}$, giving the answer $264^2 = 69696$. □

Problem 22. **[16]** Let positive integer T_n be the number of ways to tile a $2 \times n$ grid with L-shaped tiles (with rotations and reflections) and unit square tiles shown below. Compute the remainder when T_{2019} is divided by 66.

Answer. $\boxed{11}$ *Proposed by Kimi*

Proof. Starting from the leftmost vertical line moving right, we consider the first time (after k units) we can draw a vertical line that does not cut into any tile. If $k = 1$, the 2×1 must be tiled by 2 unit types, giving T_{n-1} ways for the rest; if $k = 2$, the 2×2 must be tiled by an L and a unit tile (4 ways) and T_{n-2} ways for the rest, giving $4T_{n-2}$; if $k = 3$, the 2×3 must be tiled by 2 complementing L's (2 ways) and T_{n-3} ways for the rest. Thus, we obtain recursion

$$T_n = T_{n-1} + 4T_{n-2} + 2T_{n-3}$$

where $T_0 = T_1 = 1$ and $T_2 = 5$. Starting with index 0: modulo-2, T_n is always $[1]$; modulo-3, T_n cycles $[1, 1, 2, 2, 0, 0]$; modulo-11, T_n cycles $[1, 1, 5, 0, 0, 10, 10, 6, 0, 0]$. Thus, $T_{2019} \equiv 1 \pmod 2$, $T_{2019} \equiv 2 \pmod 3$, and $T_{2019} \equiv 0 \pmod{11}$. Thus, CRT gives $T_{2019} \equiv 11 \pmod{66}$. □

Problem 23. **[17]** Compute the number of arithmetic sequence of integers $(a_n)_{n=1}^{\infty}$ with $a_1 = 2019$ that satisfies the following: for every positive integer n, there exist a positive integer m such that $\sum_{k=1}^{n} a_k = a_m$.

Answer. $\boxed{5}$ *Proposed by Kimi*

Proof. Let the common difference of the sequence be d. Then the sum of the first n terms is

$$2019 + (2019 + d) + \cdots + (2019 + d \cdot (n-1)) = 2019n + \frac{(n-1)n}{2}d = 2019 + 2019(n-1) + \frac{(n-1)n}{2}d.$$

This is a term in the sequence only if $d \mid 2019(n-1) + \frac{(n-1)n}{2}d$, meaning that $d \mid 2019(n-1)$. However, the only way this can be true for all n is if $d \mid 2019$. Thus, d must be a factor of 2019. It is easy to see that if $d > 0$ the condition in the problem is satisfied, since we are summing multiples of d. If $d < 0$ and $d \neq -2019$ then $a_1 + a_2 > 2019$, a contradiction. If $d = -2019$, the condition is also satisfied. Since 2019 has 4 divisors, there are $4 + 1 = 5$ such sequences. □

Problem 24. **[17]** Given n people, they can form a set of non-empty groups such that each person is in exactly one group; in each group of k people, they hold hands to form a single big *circle* with k people. We say two such *arrangements* are identical if one can be obtained from the other by permuting the set of circles and/or rotating each circle. Compute the number of distinct arrangements for $n = 2019$ people.

Answer. $\boxed{2019!}$ *Proposed by Kimi*

Proof. Label the people $1, 2, \ldots, n$. Use parentheses to denote grouping into circles. The example in the problem is $(1)(23)(456) \sim (1)(645)(32) \not\sim (1)(23)(546)$. We exhibit a bijection from distinct arrangements to S_n, the set of permutations of $\{1, \ldots, n\}$.

Given an arrangement, we define σ as follows: for person k, define $\sigma(k)$ to the person to the right of k, i.e. for $(\ldots, k, l, \ldots)$, take $\sigma(k) = l$. Note, take $\sigma(k) = k$ if k is in a group of 1, i.e. (k). Clearly, $\sigma \in S_n$ and distinct arrangements corresponds to distinct bijections.

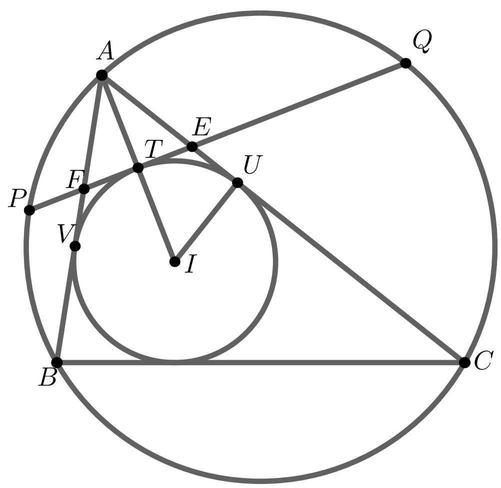
Conversely, given any $\sigma \in S_n$, we create an arrangement: the first circle is $\big(1, \sigma(1), \sigma^2(1), \ldots\big)$ where $\sigma^k = \sigma \circ \sigma^{k-1}$. For each k, if $\sigma^k(1) \notin \{1, \sigma(1), \ldots, \sigma^{k-1}(1)\}$, we can put person $\sigma^k(1)$ to the right of person $\sigma^{k-1}(1)$. We can do so until, for the first time (smallest k), some $\sigma^k(1) = \sigma^l(1)$

for $l < k$. If $l \geq 1$, since σ is a bijection, $\sigma^{l-1}(1) = \sigma^{-1}(\sigma^l(1)) = \sigma^{-1}(\sigma^k(1)) = \sigma^{k-1}(1)$. This contradicts the minimality of k unless $l = 0$, where $\sigma^0(1) = 1$. When that happens, we stop and the circle is complete as it loops back to person 1. We now pick any person j not in the circle (if any) and repeat the process with j instead of 1 until we exhaust all n people. Clearly, this is valid.

Thus, the number of distinct arrangements is $|S_n| = n!$, giving the answer 2019!. □

Remark. If we do not write down groups of one (and assume they are fixed points of σ by default), this notation is the well-known cycle notation for the symmetric group on n letters.

Problem 25. [18] In $\triangle ABC$ with incenter I, $AB = 5$, $BC = 7$, and $CA = 8$. Segment AI intersects the incircle at point T, and the line tangent to the incircle at T intersects the circumcircle of $\triangle ABC$ at P and Q. Let I_B and I_C be the B-excenter and C-excenter of $\triangle ABC$, respectively. A point X is chosen on segment I_BI_C. Compute the maximum possible area of $\triangle XPQ$.



Answer. $\boxed{\dfrac{7\sqrt{3}}{2}}$ *Proposed by Akash*

Proof. Note that I_BI_C is the external angle bisector of A, so it is perpendicular to AI. Since AI is perpendicular to PQ, we get that PQ and I_BI_C are parallel, so the area of XPQ is constant for all points X on segment I_BI_C. Thus, without loss of generality, let X be the point A.

Consider the diagram above. Note that because $AB^2 + AC^2 - 2AB \cdot BC\cos(A) = BC^2$, we can plug in our sidelengths to get $\cos(A) = \frac{1}{2}$, so $\angle A = 60°$. Note that we have $[ABC] = \sqrt{10 \cdot 2 \cdot 3 \cdot 5} = 10\sqrt{3}$, by Heron's formula. Thus, we know that the inradius is given by $r = \frac{ABC}{s}$, where s is the semi-perimeter. Plugging in $[ABC] = 10\sqrt{3}$ and $s = 10$ gives us $r = \sqrt{3}$. Since $\angle A = 60°$, we have that $\angle IAU = 30°$, so $AI = 2IU = 2\sqrt{3}$, so $AT = AI - IT = 2\sqrt{3} - \sqrt{3} = \sqrt{3}$. Since $\triangle AFE$ is equilateral, we have $AF = FE = EA = 2$. Let $x = PF$ and $y = EQ$. By Power of Point from points F and E, we get:

$$PF \cdot FQ = AF \cdot FB \implies x(2 + y) = 2 \cdot 3 = 6$$
$$PE \cdot EQ = AE \cdot EC \implies y(2 + x) = 2 \cdot 6 = 12$$

Subtracting the first from the second gives $2y - 2x = 12 - 6 = 6$, so $y = x + 3$. Plugging this into the first one equation gives $x(5 + x) = 6$, so $x = 1$, and thus $y = 4$. Hence, we have $PQ = PF + FE + EQ = 1 + 2 + 4 = 7$. Thus, the area of APQ is $\frac{1}{2}AT \cdot PQ = \frac{7\sqrt{3}}{2}$ □

Problem 26. **[18]** $n = 2010^3 + 600^3 + 67^6$ is the product of a three-digit prime number, a four-digit prime number, and a five-digit prime number. Find the four-digit prime factor of n.

Answer. $\boxed{3079}$ *Proposed by Rishabh*

Proof. Let $x = 2010 = 2 \cdot 3 \cdot 5 \cdot 67, y = 600 = 2^3 \cdot 3. \cdot 5^2$, and $z = 67^2$. Then note that $3yz = 2x^2$. Then:

$$n = x^3 + y^3 + z^3 = (-x)^3 + y^3 + z^3 + 2x^3 = (-x)^3 + y^3 + z^3 - 3(-x)yz.$$

Using the well-known factorization $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$, we know that $y + z - x$ is a factor of n. $y + z - x = 600 + 67^2 - 2010 = 3079$, which must be prime since otherwise n would have a prime factor with less than 3 digits. □

Problem 27. **[19]** Compute

$$\sum_{n=1}^{\infty} \frac{(-2)^{2^n+n}}{2^{2^{n+1}} - 2^{2^n} + 1}$$

Answer. $\boxed{-\frac{8}{21}}$ *Proposed by Akash*

Proof. Let $f(n) = \dfrac{2^{2^n+n}}{2^{2^{n+1}} - 2^{2^n} + 1}$ and $g(n) = \dfrac{2^{2^n+n}}{2^{2^{n+1}} + 2^{2^n} + 1}$. Note that

$$\begin{aligned} g(n) - f(n) &= \frac{2^{2^n+n}}{2^{2^{n+1}} + 2^{2^n} + 1} - \frac{2^{2^n+n}}{2^{2^{n+1}} - 2^{2^n} + 1} \\ &= \frac{2^{2^n+n}(2^{2^{n+1}} - 2^{2^n} + 1) - 2^{2^n+n}(2^{2^{n+1}} + 2^{2^n} + 1)}{(2^{2^{n+1}} - 2^{2^n} + 1)(2^{2^{n+1}} + 2^{2^n} + 1)} \\ &= -\frac{2^{2^n+n} \cdot 2^{2^n+1}}{2^{2^{n+2}} + 2^{2^{n+1}} + 1} \\ &= -\frac{2^{2^{n+1}+n+1}}{2^{2^{n+2}} + 2^{2^{n+1}} + 1} \\ &= -g(n+1). \end{aligned}$$

Also note

$$\frac{(-2)^{2^n+n}}{2^{2^{n+1}} - 2^{2^n} + 1} = (-1)^n f(n).$$

Let S_n be the nth partial sum of the sum we want. Then:

$$\begin{aligned} g(1) + S_n &= g(1) - f(1) + f(2) - f(3) + \cdots + (-1)^n f(n) \\ &= -g(2) + f(2) - f(3) + \cdots + (-1)^n f(n) \\ &= g(3) - f(3) + \cdots + (-1)^n f(n) \\ &\vdots \\ &= (-1)^n g(n+1) \end{aligned}$$

As n goes to infinity, $(-1)^n g(n+1)$ goes to 0. Thus, as n goes to infinity, S_n goes to $-g(1)$, so the original sum goes to $-g(1) = -\frac{8}{21}$. □

Problem 28. **[19]** 2019 numbers are chosen uniformly at random from the range $[0,1]$. Given that the largest of the numbers is at least $\frac{1}{5}$ larger than all other numbers, what is the expected value of the largest number?

Answer. $\boxed{\dfrac{2524}{2525}}$ *Proposed by Akash*

Proof. Given such 2019 numbers $0 \le x_1 \le x_2 \le \cdots \le x_{2019} \le 1$ where $x_{2019} - x_{2018} \ge \frac{1}{5}$, we take $y_i = x_i$ for $1 \le i \le 2018$ and $y_{2019} = x_{2019} - \frac{1}{5}$ to biject it to 2019 numbers in $[0, \frac{4}{5}]$. Again, we biject it to 2020 positive reals whose sum is $\frac{4}{5}$ by taking $z_i = y_i - y_{i-1}$ for $1 \le i \le 2019$ and $z_{2020} = \frac{4}{5} - y_{2019}$. By symmetry, $\mathbb{E}(z_{2020}) = \frac{4}{5} \cdot \frac{1}{2020} = \frac{1}{2525}$. Finally, linearity of expectation gives

$$\mathbb{E}\left(x_{2019}\right) = \mathbb{E}\left(y_{2019} + \frac{1}{5}\right) = \mathbb{E}\left(\left(\frac{4}{5} - z_{2020}\right) + \frac{1}{5}\right) = 1 - \mathbb{E}\left(z_{2020}\right) = \frac{2524}{2525} \quad \square$$

Problem 29. **[20]** Let O, I, and H denote the circumcenter, incenter, and orthocenter of $\triangle ABC$. Given that $OI = \sqrt{901}, OH = 3\sqrt{401}$, and $HI = 2\sqrt{226}$, compute the sum of the inradius and circumradius of $\triangle ABC$.

Answer. $\boxed{1351}$ *Proposed by Rishabh*

Proof. Let R and r denote the circumradius and inradius of $\triangle ABC$. Let N be the midpoint of OH, which is also the center of the nine-point circle. From the median formula, we see

$$IN = \frac{1}{2}\sqrt{2IO^2 + 2IH^2 - OH^2} = \frac{1}{2}\sqrt{2 \cdot 901 + 2 \cdot 904 - 3609} = \frac{1}{2}$$

By Feurbach's Theorem, the nine-point circle is tangent to the incircle. Since the nine-point circle has radius $\frac{R}{2}$, we know that $IN = \left|\frac{R}{2} - r\right|$. However, by Euler's Inequality, $R \ge 2r$, so actually $IN = \frac{R}{2} - r$. Thus, $R - 2r = 1$. Finally, Euler's Formula says that $OI^2 = R(R-2r) = R$, so $R = 901$. This means $r = 450$, so $R + r = 1351$. $\square$

Problem 30. **[20]** For positive real $y \neq 2$, find the product of all possible positive real x as a function of y such that

$$\sqrt{x+4} + \sqrt{y+2} + \sqrt{y+4} + 2 = \sqrt{\left(\sqrt{x+4}+2\right)\left(\sqrt{y+2}+2\right)\left(\sqrt{y+4}+2\right)}$$

Answer. $\boxed{4}$ *Proposed by Ethan*

Proof. Let positive reals x, y, z satisfy $x^2 + y^2 + z^2 - xyz = 4$ and $\max(x,y,z) > 2$ then there exist reals a, b, c such that $abc = 1$ and $x = a + \frac{1}{a}$, $y = b + \frac{1}{b}$, $z = c + \frac{1}{c}$. Prove this by expanding and quadratic formula. So since $\sqrt{a + 2 + \frac{1}{a}} = \sqrt{a} + \frac{1}{\sqrt{a}}$, x, y, z work for the above if and only if $\sqrt{x+2}, \sqrt{y+2}, \sqrt{z+2}$ work. Because $\max(x+2, y, y+2) \ge (y+2) > 2$, $(x+2)^2 + y^2 + (y+2)^2 - (x+2)y(y+2) = 4$ if and only if $(x+2+2) + (y+2) + (y+2+2) - \sqrt{(x+2+2)(y+2)(y+2+2)} = 4$ if and only if $\sqrt{x+4} + \sqrt{y+2} + \sqrt{y+4} + 6 - \sqrt{\left(\sqrt{x+4}+2\right)\left(\sqrt{y+2}+2\right)\left(\sqrt{y+4}+2\right)} = 4$. Which is equivalent to the condition we are given. So the solutions are the solutions to $(x+2)^2 + y^2 + (y+2)^2 - (x+2)y(y+2) = 4$. The product of the roots is 4 by Vieta since $y \neq 2$ so there can't be double roots. Note that since $x + 2 = a + \frac{1}{a}$, $x \ge 0$ and $x \neq 0$ since the product is not 0. $\square$