Stuyvesant Team Contest

Fall 2019

1. **[6]** Compute the smallest positive integer n such that if n students participate in the Stuyvesant Team Contest, they can be split evenly into $1, 2, 3, 4, 5, 6,$ and 7 teams.

   Team Name: ____________ Answer: ____________

2. **[6]** $ABCD$ is a trapezoid has area 276 and $AB \parallel CD$. Points M and N are midpoints of segment AD and BC, respectively. Compute the area of quadrilateral $DMBN$.

   Team Name: ____________ Answer: ____________

3. **[7]** Given that real number r satisfies $|2019 - r| + \sqrt{r - 2020} = r$, compute all possible values of $r - 2019^2$.

   Team Name: ____________ Answer: ____________

4. **[7]** How many distinct 8 digit numbers can be formed by concatenating exactly one of each of $\{2, 0, 1, 9, 2019\}$?

   Team Name: ____________ Answer: ____________

5. **[8]** If x and y are positive reals such that:

$$x + y^2 = 2019$$
$$x^2 + y^2 = 2109$$

   then compute $x^3 + y^2$

   Team Name: ____________ Answer: ____________

6. **[8]** Let $f(x) = \frac{x}{\sqrt{1+x^2}}$ and $f_n(x) = \underbrace{f(f(\dots f}_{n \text{ times}}(x)\dots))$, compute $f_{99}(1)$.

   Team Name: ____________ Answer: ____________

7. **[9]** Equilateral $\triangle ABC$ has side-length 2019. Points $X, Y,$ and Z are on segments BC, CA, and AB, respectively. If $CX = CY = BZ = 673$, compute the radius of the circle passing through $X, Y,$ and Z.

   Team Name: ____________ Answer: ____________

8. **[9]** A soccer ball is glued together edge-to-edge from 32 polygons, each of which is either a pentagon or a hexagon. Given each pentagon is glued to 5 hexagons and each hexagon is glued to 3 pentagons and 3 hexagons, compute the number of hexagons.

Team Name: ______________ Answer: ______________

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9. **[10]** Compute the sum of all possible non-negative integers n such that $0! + 1! + \cdots + n!$ is a perfect square. *Note*: $0! = 1$.

Team Name: ______________ Answer: ______________

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10. **[10]** The number of teams is N. Submit an integer a between 0 and N, inclusive. Let A be the average of all submissions and n be the number of submissions greater than A. You will receive $\left\lceil \frac{20}{2+|a-n|} \right\rceil$ points.

Team Name: ______________ Answer: ______________

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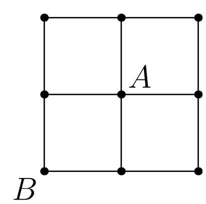
11. **[11]** An ant starts at point A. Every second, picks a point that it is adjacent to at random, and moves to this point. (Adjacent means connected by an edge.) What is the probability that after 2020 seconds, the ant is on point B?



Team Name: ______________ Answer: ______________

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12. **[11]** Let $AB = 20, BC = 29$, and $CA = 21$. Reflect A over BC to get A'. Reflect A' over AB and AC to get X and Y, respectively. Find the area of quadrilateral $XYCB$.

Team Name: ______________ Answer: ______________

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13. **[12]** Compute the number of pairs of positive integers (m, n) such that $m, n \leq 30$ and

$$m + \gcd(m, n) = n + \operatorname{lcm}(m, n)$$

Team Name: ______________ Answer: ______________

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14. **[12]** If a is selected from $\{1, 2, \ldots, 10\}$ uniformly randomly and b is independently (of a) selected from $\{-10, -9, \ldots, -1\}$ uniformly randomly, compute the probability $a^2 + b$ is a multiple of 3.

Team Name: ______________ Answer: ______________

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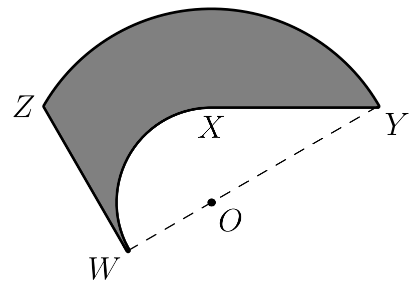
15. **[13]** In the diagram below, arcs WX and YZ both have center O with radii 1 and 2. Given $\angle OXY = \angle OWZ = 90°$ and points W, O, Y are collinear, compute the area of the shaded region $WXYZ$.



Team Name: ______________ Answer: ______________

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16. **[13]** The roots of $x^3 - 14x^2 + 54x - p = 0$ are positive real numbers that form a right triangle. Find p.

Team Name: ______________ Answer: ______________

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17. **[14]** Point D is on side BC of $\triangle ABC$ such that $AD \perp BC$, $AB + BD = DC$, and $\angle B = 40°$. Compute $\angle C$.

Team Name: ______________ Answer: ______________

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18. **[14]** Find the last two digits of the sum of all positive integers x such that $(\sqrt{x} + \sqrt{x + 2019})^2$ is an integer.

Team Name: ______________ Answer: ______________

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19. **[15]** Real numbers x and y satisfy $-\frac{\pi}{2} < y < 0 < x < \frac{\pi}{2}$. Given $\sin y + \tan^2 x = \sin x + \tan^2 y$ and $\sin^2 x + 2\cos(x - y) + \sin^2 y = \frac{17}{16}$, compute $\sin x + \sin y$.

Team Name: ______________ Answer: ______________

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20. **[Up to 64]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 *letters corresponding to each question: put* T *if you think the statement is true,* F *if you think it is false, and* X *if you do not wish to answer. You will receive* 2^n *points for* n *correct answers, but you will receive* 0 *if any of the questions you choose to answer is answered incorrectly.*

(1) Define a magic square to be a 3-by-3 square of distinct numbers such that all rows, columns, and the two diagonals diagonal have the same sum. Define a unit fraction as a fraction of the form $\frac{1}{n}$ for a positive integer n. There exists a magic square consisting of only unit fractions.

(2) A knight's tour is a sequence of moves of a knight on a chessboard such that the knight visits every square only once. If the knight ends on a square that is one knight's move from the beginning square (so that it could tour the board again immediately, following the same path), the tour is closed. There exists a closed knight's tour on a 4×4 chessboard.

(3) There exists a closed two-dimensional shape with three non-concurrent lines of symmetry.

(4) Call a function $f : \mathbb{R} \to \mathbb{R}$ *goofy* if $|f(a) - f(b)|^{2019} \leq |a - b|^{2020}$ for all reals a and b. Then, every *goofy* function must be a constant function.

(5) Let $\mathcal{S}$ be the set of positive integers with no two consecutive digits that sum up to 9. The, the sum

$$\sum_{x \in \mathcal{S}} \frac{1}{x}$$

diverges, i.e. for every M, there exist finite subset $\mathcal{T} = \{t_1, t_2, \ldots, t_n\} \subset \mathcal{S}$ such that $\sum_{k=1}^{n} \frac{1}{t_k} \geq M$.

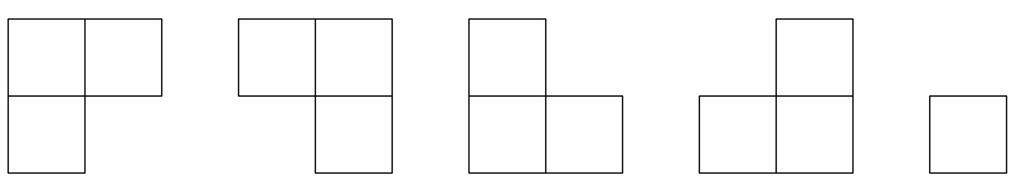
(6) In regular tetrahedron $ABCD$, if points X and Y are chosen on faces ABC and BCD, then there must exists a triangle with side lengths AY, DX, and XY.

Team Name: _____________ Answer: _____________

21. **[16]** For any positive three digit number, define its *Jerry's* to be the set of all distinct two digit integers formed with exactly one copy of each of its digits. For example, the Jerry's of 669 are $\{66, 96, 69\}$; and the Jerry's of 420 are $\{20, 40, 24, 42\}$. Let $\mathcal{J}$ be the set of all positive three digit numbers equal to the sum of its Jerry's. Compute the product of the mean of $\mathcal{J}$ and the median of $\mathcal{J}$.

Team Name: _____________ Answer: _____________

22. **[16]** Let positive integer T_n be the number of ways to tile a $2 \times n$ grid with L-shaped tiles (with rotations and reflections) and unit square tiles shown below. Compute the remainder when T_{2019} is divided by 66.



Team Name: _____________ Answer: _____________

23. **[17]** Compute the number of arithmetic sequence of integers $(a_n)_{n=1}^{\infty}$ with $a_1 = 2019$ that satisfies the following: for every positive integer n, there exist a positive integer m such that $\sum_{k=1}^{n} a_k = a_m$.

Team Name: _____________ Answer: _____________

24. **[17]** Given n people, they can form a set of non-empty groups such that each person is in exactly one group; in each group of k people, they hold hands to form a single big *circle* with all k people. We say two such *arrangements* are identical if one can be obtained from the other by permuting the set of circles and/or rotating each circle. Compute the number of distinct arrangements for $n = 2019$ people.

   For example, for $n = 6$, arrangement {Me, Milan and Matthew, Max and Mario and Maxwell} is identical to {Me, Maxwell and Max and Mario, Matthew and Milan}; but not identical to {Me, Milan and Matthew, Mario and Max and Maxwell}.

Team Name: ____________ Answer: ____________

25. **[18]** In $\triangle ABC$ with incenter I, $AB = 5$, $BC = 7$, and $CA = 8$. Segment AI intersects the incircle at point T, and the line tangent to the incircle at T intersects the circumcircle of $\triangle ABC$ at P and Q. Let I_B and I_C be the B-excenter and C-excenter of $\triangle ABC$, respectively. A point X is chosen on segment $I_B I_C$. Compute the maximum possible area of $\triangle XPQ$.

Team Name: ____________ Answer: ____________

26. **[18]** $n = 2010^3 + 600^3 + 67^6$ is the product of a three-digit prime number, a four-digit prime number, and a five-digit prime number. Find the four-digit prime factor of n.

Team Name: ____________ Answer: ____________

27. **[19]** Compute

$$\sum_{n=1}^{\infty} \frac{(-2)^{2^n+n}}{2^{2^{n+1}} - 2^{2^n} + 1}$$

Team Name: ____________ Answer: ____________

28. **[19]** 2019 numbers are chosen uniformly at random from the range $[0, 1]$. Given that the largest of the numbers is at least $\frac{1}{5}$ larger than all other numbers, what is the expected value of the largest number?

Team Name: ____________ Answer: ____________

29. **[20]** Let O, I, and H denote the circumcenter, incenter, and orthocenter of $\triangle ABC$. Given that $OI = \sqrt{901}, OH = 3\sqrt{401}$, and $HI = 2\sqrt{226}$, compute the sum of the inradius and circumradius of $\triangle ABC$.

Team Name: ____________ Answer: ____________

30. **[20]** For positive real $y \neq 2$, find the product of all possible positive real x as a function of y such that

$$\sqrt{x+4} + \sqrt{y+2} + \sqrt{y+4} + 2 = \sqrt{\left(\sqrt{x+4}+2\right)\left(\sqrt{y+2}+2\right)\left(\sqrt{y+4}+2\right)}$$

Team Name: ____________ Answer: ____________