2018 Stuyvesant Team Contest Solutions

1. [**5**] Compute:

$$\left(\frac{1}{6}\right)^3+\left(\frac{1}{3}\right)^3+\left(\frac{1}{2}\right)^3$$

Solution

$$\left(\frac{1}{6}\right)^3+\left(\frac{1}{3}\right)^3+\left(\frac{1}{2}\right)^3=\frac{1^3+2^3+3^3}{6^3}=\frac{1+8+27}{6^3}=\frac{36}{6^3}=\frac{6^2}{6^3}=\boxed{\frac{1}{6}}$$

2. [**5**] How many non-congruent, non-degenerate triangles have integer side lengths and perimeter less than 8?

Solution

Let the sides of the triangle be $a \le b \le c$. So we must have that $a+b+c<7$ and $a+b>c$. We make a table to list all of the possible triangles:

a	b	$a+b$	c	$a+b+c$
1	1	2	1	3
1	2	3	2	5
1	3	4	3	7
2	2	4	2	6
2	2	4	3	7

Thus the answer is $\boxed{5}$.

3. [**5**] Compute the number of nonnegative integers n such that $3^n \ge n!$.

Note: $n!$ is defined as the product of all the positive integers less than or equal to n. An empty product is equal to 1.

Solution

By computation, we find that $n=0,1,\dots,6$ all work. However, for $n=7$, $3^n<n!$. Additionally, since $n!$ grows faster than 3^n for $n>3$, there are only $\boxed{7}$ values of n that work.

4. [**6**] Let a,b,c be reals such that $\frac{a+b}{c}=1$ and $\frac{b+c}{a}=2$. Compute $\frac{c+a}{b}$.

Solution 1

We add 1 to both equations giving us:

$$\frac{a+b+c}{c}=2,\ \frac{a+b+c}{a}=3$$

If we take the reciprocals of both equations and subtract them from 1, we get:

$$\frac{b}{a+b+c}=1-\frac{c}{a+b+c}-\frac{a}{a+b+c}=1-\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$$

We take reciprocals and subtract 1:

$$\frac{c+a}{b}=\frac{a+b+c}{b}-\frac{b}{b}=6-1=\boxed{5}$$

Solution 2

Let $s=a+b+c$. Then

$$\frac{s-c}{c}=\frac{a+b}{c}=1,\ \frac{s-a}{a}=\frac{b+c}{a}=2$$

So, $s-c=c$, or $c=\frac{s}{2}$. Similarly, we find that $a=\frac{s}{3}$. So, $b=s-a-c=\frac{s}{6}$. Thus

$$\frac{c+a}{b}=\frac{\frac{s}{2}+\frac{s}{3}}{\frac{s}{6}}=\boxed{5}$$

5. [**6**] If a and b are positive integers such that $2a^2 = 3b^3$, compute the minimum possible value of $a + b$.

   Solution

   Note that a must be a multiple of 3 and b must be a multiple of 2. So, let $a = 3c$ and $b = 2d$, for positive integers c and d. Substituting into the original equation, we get: $18c^2 = 24d^3$. Dividing by 6, we get: $3c^2 = 4d^3$. So, c must be a multiple of 2 and d must be a multiple of 3. Let $c = 2e$ and $d = 3f$, for positive integers e and f. Substituting into the previous equation, we get: $12e^2 = 108f^3$. Dividing by 12, we get $e^2 = 9f^3$. Now, e must be a multiple of 3 so we let $e = 3g$ for a positive integer g. This leads to $g^2 = f^3$. To minimize a and b, we minimize f and g. So, let $f = g = 1$, which gives $a = 18$ and $b = 6$. Thus the answer is $18 + 6 = \boxed{24}$.

6. [**6**] Three distinct lines are drawn in the plane. One pair forms an angle of 83°. Another pair forms an angle of 97°. The third pair forms an angle of n°. Compute smallest possible value of n.

   Solution

   Consider the triangle formed by the three lines. Since the lines are distinct, the triangle is non-degenerate. Thus all of its angles are positive. One angle of this triangle is either 83° or its supplement. Another angle is either 97° or its supplement. The third angle is either n° or its supplement. The only way for the angles to each be positive and add to 180° is to have the first two angles be 83° and the supplement of 97°, which is also 83°. Thus either $n^\circ = 180^\circ - 2 \cdot 83^\circ$ or $180^\circ - n^\circ = 180^\circ - 2 \cdot 83^\circ$. Thus, the smallest possible value of n is $\boxed{14}$.

7. [**7**] How many subsets of the set $\{1, 2, 4, 8, 16, -1, -2, -4, -8, -16\}$ have a sum of 0?

   Note: The empty set has a sum of 0.

   Solution

   Let $S = \{1, 2, 4, 8, 16, -1, -2, -4, -8, -16\}$. For each subset T of $\{1, 2, 4, 8, 16\}$, we can take all of the elements of T and their negatives to get a subset of S with a sum of 0. Additionally, for each integer from 0 to 31 inclusive there is exactly one way to write it as a sum of elements of such a set T. This accounts for all possible sums of subsets of $\{1, 2, 4, 8, 16\}$. Therefore, once we have chosen which elements of $\{1, 2, 4, 8, 16\}$ to include in the subset of S, we are forced to include exactly their negatives to get a sum of 0. Thus the answer is exactly the number of subsets of $\{1, 2, 4, 8, 16\}$, which is $2^5 = \boxed{32}$.

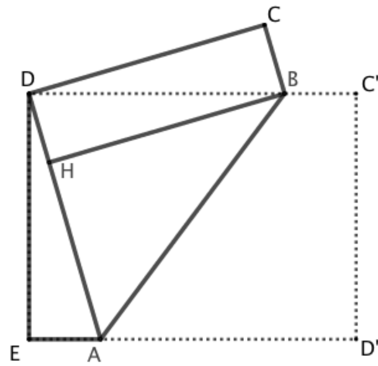
8. [**7**] How many distinct values does $\lfloor x \rfloor + \lceil x \rceil$ take for x in the interval $(-2018, 2018)$?

   Solution

   When x is an integer, $\lfloor x \rfloor + \lceil x \rceil = 2x$. There are 4035 integers in the interval $(-2018, 2018)$, so this gives 4035 distinct even values for $\lfloor x \rfloor + \lceil x \rceil$. When x is not an integer $\lfloor x \rfloor + \lceil x \rceil = \lfloor x \rfloor + \lfloor x \rfloor + 1 = 2\lfloor x \rfloor + 1$. For x in the interval $(-2018, 2018)$, $\lfloor x \rfloor$ can range from -2018 to 2017, with 4036 distinct values. Each of these values gives a distinct odd value for $\lfloor x \rfloor + \lceil x \rceil$. Thus there is no overlap between the two cases and the answer is $4035 + 4036 = \boxed{8071}$.

9. [**7**] Stan takes a rectangular 24 by 32 sheet of paper and folds one of its corners onto the opposite corner. Compute the area of the pentagon formed by the folded paper.

   Diagram



Solution 1

Let the pentagon be $ABCDE$ as shown. Since $ABCDE$ is the same paper as the original rectangle with $\triangle ABD$ covered twice,

$$[ABCDE] = [DED'C'] - [ABD]$$

We have $[DED'C'] = 24 \cdot 32 = 768$. Now we compute $[ABD]$.

Note that $AD + AE = AD' + AE = ED' = 32$. Moreover, since $\triangle ADE$ is a right triangle with AD as the hypotenuse, we can use the Pythagorean Theorem: $AD^2 = AE^2 + DE^2 = AE^2 + 24^2$. We can solve this system of equations to find $AD = 25$ and $AE = 7$. Let the altitude from B to AD intersect AD at H. Since $DHBC$ is a rectangle and by symmetry, $BH = CD = AD = 24$. This makes $[ABD] = \frac{1}{2} \cdot 25 \cdot 24 = 300$. Thus the answer is $768 - 300 = \boxed{468}$.

Solution 2

Note that

$$[ABCDE] = [ABCD] + [AED] = [ABC'D'] + [AED] = \frac{1}{2}[DED'C'] + [AED]$$

As in Solution 1, we compute $[DED'C'] = 768$ and $AE = 7$. Then the answer is $\frac{1}{2} \cdot 768 + \frac{1}{2} \cdot 7 \cdot 24 = \boxed{468}$.

10. [**8**] Let $d(n)$ denote the number of positive divisors of n. Compute the largest integer $n < 100$ such that

$$d(d(d(n))) = d(n)$$

Solution

Note that $d(n) \le n$ with equality only when $n = 1$ or $n = 2$. So, $d(d(d(n))) \le d(d(n)) \le d(n)$. Since $d(d(d(n))) = d(n)$, we must have $d(n) = 2$ or $d(n) = 1$. This only occurs when $n = 1$ or n is prime. Thus the largest value of n is the largest prime less than 100, which is $\boxed{97}$.

11. [**8**] Compute the number of triples of positive integers (a, b, c) for which $18! \mid a \mid b \mid c \mid 20!$.

Note: We write $m \mid n$ if m is a factor of n.

Solution

Let $p = \frac{a}{18!}$, $q = \frac{b}{a}$, $r = \frac{c}{b}$, and $s = \frac{20!}{c}$. Then p, q, r, and s are positive integers with

$$pqrs = \frac{20!}{18!} = 19 \cdot 20 = 2^2 \cdot 5 \cdot 19$$

For each such choice of p, q, r, and s we get an ordered triple (a, b, c). Thus we just need to count the number of ordered quadruples of positive integers (elements) with product $2^2 \cdot 5 \cdot 19$. There are 4 ways to distribute both factors of 2 to the same element, and $\binom{4}{2} = 6$ ways to distribute the factors of 2 to different elements. So, we have $4 + 6 = 10$ ways to distribute the factors of 2 between the elements. There are 4 ways to distribute the 5 and 4 ways to distribute the 19 to the elements. Thus the answer is $10 \cdot 4 \cdot 4 = \boxed{160}$.

12. [**8**] In $\triangle ABC$, $AB = 13$, $BC = 14$, and $AC = 15$. Points X, Y, and Z are the trisection points of sides BC, AC, and AB, respectively with X closer to B, Y closer to C, and Z closer to A. Let A', B', and C' be the reflections of A, B, and C over X, Y, and Z, respectively. Compute $[A'BC'AB'C]$.

    Note: $[\cdots]$ denotes the area of the polygon in the brackets.

    Solution

    Note that X is the midpoint of AA'. Thus $AX = A'X$ and the altitudes from A and A' to BC are equal. Then the bases and heights of $\triangle ABC$ and $\triangle A'BC$ are equal, so their areas are equal. Similarly, $[ABC] = [A'BC] = [AB'C] = [ABC']$. Now, we have that

$$[A'BC'AB'C] = [ABC] + [A'BC] + [AB'C] + [ABC'] = 4\,[ABC] = 4 \cdot 84 = \boxed{336}$$

13. [**9**] Matthew flips 2018 fair coins. Milan flips 2017 fair coins. Compute the probability that Matthew gets more heads than Milan.

    Solution 1

    Say that Matthew wins if he gets more heads than Milan. Consider a game where Milan starts with $\frac{1}{2}$ of a point, and both players get 1 point for each heads they flip. Matthew will get more points if and only if he gets more heads than Milan, so this game is equivalent to the original problem. In this game, both players' scores are symmetric about 1009, and there cannot be a tie. So, we can pair up the outcomes where Matthew with the outcomes where he loses. Thus the answer is $\boxed{\frac{1}{2}}$.

    Solution 2

    Suppose that both Matthew and Milan had 2017 coins. In this case, let p_{2017} be the probability that Matthew gets more heads than Milan. Note that $1 - p_{2017}$ is the probability that Milan gets at least as many heads as Matthew, which by symmetry is also the probability that Matthew gets at least as many heads as Milan.

    In the original problem, we consider cases based on Matthew's first flip. If he gets a heads, then he needs at least as many heads as Milan in the remaining 2017 flips. If he gets a tails, then he needs more heads than Milan in the remaining 2017 flips. Each of these cases has probability $\frac{1}{2}$ of occurring, so the answer is $\frac{1}{2} \cdot (1 - p_{2017}) + \frac{1}{2} \cdot p_{2017} = \boxed{\frac{1}{2}}$.

14. [**9**] A sequence $\{a_n\}$ is defined for positive integers n so that $a_1 = a_2 = 1$ and for all $n > 2$,

$$a_n = \frac{a_{n-1} \cdot a_{n-2}}{a_{n-1} + a_{n-2}}$$

    Compute a_{12}.

    Solution

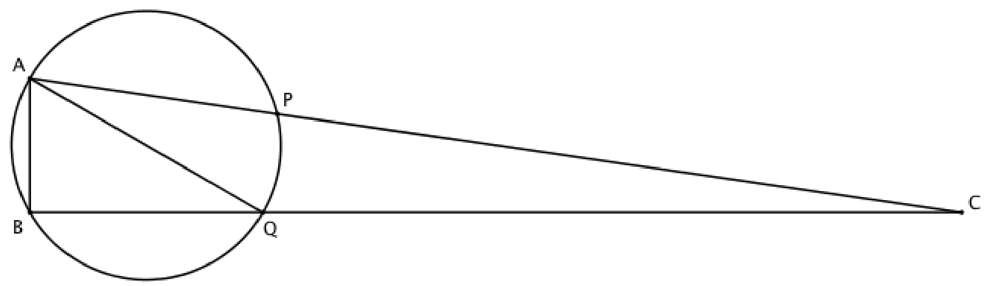
    Taking the reciprocal of both sides of the given equation, we get:

$$\frac{1}{a_n} = \frac{1}{a_{n-1}} + \frac{1}{a_{n-2}}$$

    Let $b_n = \frac{1}{a_n}$. Then $b_n = b_{n-1} + b_{n-2}$ and $b_1 = b_2 = 1$. So, the sequence $\{b_n\}$ is just the Fibonacci sequence. We can compute $b_{12} = 144$, which gives $a_{12} = \boxed{\frac{1}{144}}$.

15. [**9**] In right $\triangle ABC$, $AB = 1$, $BC = 4\sqrt{3}$, $AC = 7$. A circle is drawn through A and B intersecting segment AC at point P such that minor arc AB has measure $60°$. Compute CP.

    Diagram



Solution

Let the circle intersect BC at Q. Since arc AB has measure $60°$, $\angle AQB = \frac{1}{2} \cdot 60° = 30°$. Also, $\angle ABQ = \angle ABC = 90°$. By $30° - 60° - 90°$ triangle properties, $BQ = \sqrt{3}$. So, $CQ = 3\sqrt{3}$. By Power of a Point, $CA \cdot CP = CB \cdot CQ$. So, $7CP = 4\sqrt{3} \cdot 3\sqrt{3} = 36$. Thus $CP = \boxed{\dfrac{36}{7}}$.

16. [**10**] Solve for all real x that satisfy:

$$\sqrt{x} - \sqrt[3]{x} + \sqrt[4]{x} = x^2 - x^3 + x^4$$

Solution

We rearrange the equation to get:

$$x^4 - x^3 + x^2 - \sqrt{x} + \sqrt[3]{x} - \sqrt[4]{x} = 0$$

Now, suppose $x > 1$. Then $x^4 > x^3$, $x^2 > \sqrt{x}$, and $\sqrt[3]{x} > \sqrt[4]{x}$. So, $x^4 - x^3 + x^2 - \sqrt{x} + \sqrt[3]{x} - \sqrt[4]{x} > 0$, which is not possible.

Similarly, if $0 < x < 1$, then $x^4 < x^3$, $x^2 < \sqrt{x}$, and $\sqrt[3]{x} < \sqrt[4]{x}$. So, $x^4 - x^3 + x^2 - \sqrt{x} + \sqrt[3]{x} - \sqrt[4]{x} < 0$, which is not possible.

Additionally, because of the square root and fourth root, we must have $x \geq 0$. Thus the only possible solutions are $x = \boxed{0, 1}$.

17. [**10**] Compute the prime factorization of $82^3 + 1$.

Solution

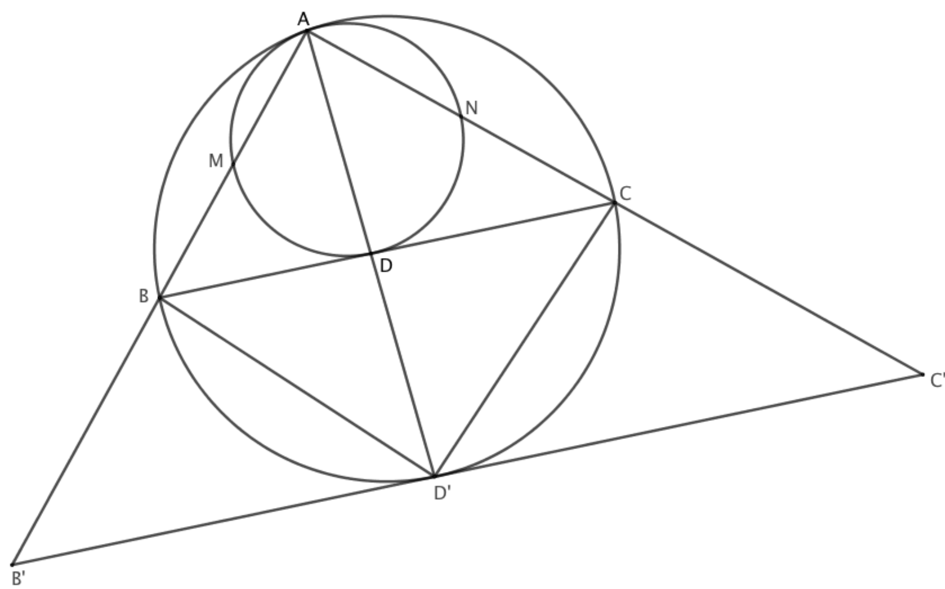
Let $x = 9$ so that $82 = x^2 + 1$ and $82^3 + 1 = (x^2 + 1)^3 + 1$. Factoring a sum of cubes and a difference of squares, we get:

$$(x^2+1)^3+1 = (x^2+1+1)((x^2+1)^2-(x^2+1)+1) = (x^2+2)((x^2+1)^2-x^2) = (x^2+2)(x^2+1+x)(x^2+1-x)$$

Substituting in $x = 9$, we get $82^3 + 1 = (9^2 + 2)(9^2 + 1 - 9)(9^2 + 1 + 9) = 83 \cdot 73 \cdot 91 = \boxed{7 \cdot 13 \cdot 73 \cdot 83}$ (Any order of the prime factors is acceptable).

18. [**10**] In $\triangle ABC$, M and N are the midpoints of AB and AC. $AB = 8$ and $AC = 9$. The circumcircle of $\triangle AMN$ is tangent to BC at D. Compute AD.

Diagram



Solution 1

We apply Power of a Point to get that $BD^2 = BM \cdot BA = \frac{1}{2} \cdot 8 \cdot 8$. So $BD = \frac{8}{\sqrt{2}}$. Similarly, $CD = \frac{9}{\sqrt{2}}$. Now we can apply Stewart's Theorem on $\triangle ABC$ with cevian AD to find AD. Alternatively, note that since $AB : AC = BD : CD$, AD must be an angle bisector. Then we can use the Second Angle Bisector Theorem to get that

$$AD^2 = AB \cdot AC - BD \cdot CD = 8 \cdot 9 - \frac{8}{\sqrt{2}} \cdot \frac{9}{\sqrt{2}} = 36$$

So, $AD = \boxed{6}$.

Solution 2

Consider a homothety (dilation) with scale factor 2 about A. This sends M to B, N to C, and D to a point D' on the circumcircle of $\triangle ABC$. Additionally, $DD' = AD$. By using Power of a Point on both circles several times, we have:

$$(DD' \cdot DA)^2 = (DB \cdot DC)^2 = BD^2 \cdot CD^2 = (BM \cdot BA) \cdot (CN \cdot CA)$$

However, we also have that $BM = \frac{BA}{2}$ and $CN = \frac{CA}{2}$. Thus,

$$AD^4 = (DD' \cdot DA)^2 = (BM \cdot BA) \cdot (CN \cdot CA) = \frac{AB^2}{2} \cdot \frac{AC^2}{2} = \left(\frac{AB \cdot AC}{2}\right)^2$$

So, $AD = \sqrt{\frac{AB \cdot AC}{2}} = \sqrt{\frac{8 \cdot 9}{2}} = \boxed{6}$.

Solution 3

Consider the same homothety as in Solution 2. Note that B and C will get sent to points B' and C' outside the circumcircle of $\triangle ABC$ such that $B'D'C'$ is collinear and tangent to the circumcircle of $\triangle ABC$ at D'. Additionally, because scaling preserves parallel lines we have that $BC \parallel B'C'$. So, $\angle B'D'B = \angle D'BC$. By inscribed angles,

$$\angle BAD' = \angle B'D'B = \angle D'BC = \angle D'AC = \angle DAC$$

Additionally, $\angle AD'B = \angle ACB = \angle ACD$. So, $\triangle BAD' \sim \triangle DAC$. Therefore, $\frac{AB}{AD'} = \frac{AD}{AC}$. Since $AD' = 2AD$, we can solve for AD: $AD = \sqrt{\frac{AB \cdot AC}{2}} = \sqrt{\frac{8 \cdot 9}{2}} = \boxed{6}$.

19. [**11**] How many proper divisors of 10! are a product of a perfect square and a perfect cube (not necessarily distinct).

Solution

Let p be a prime dividing 10!. The exponent of p in a divisor of 13! that is a product of a perfect square and cube must be of the form $2x+3y$, where $2x$ is the number of times p divides into the square and $3y$ is the number of times p divides into a cube $(x, y \geq 0)$. We note that $2x+3y$ can take on any positive integer greater than or equal to 2 and zero. Thus, if the exponent of p in 10! is α, there are α choices for what exponent we can pick, namely, $0, 2, 3, \dots, \alpha$. Since the choice of an exponent for one prime is independent from the choice of another, we simply take the product of the exponents of 10!. We factor $10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7$, which gives us an answer of $8 \cdot 4 \cdot 2 \cdot 1 = \boxed{64}$.

20. [**11**] Compute the sum of the sum of the digits of all three digit numbers with at least one digit that is a 7.

Solution

We will find the sum over all three digit numbers and subtract the sum over all three digit numbers with no digit that is a 7. For the first case, the average value of the first digit is $\frac{1+2+3+4+5+6+7+8+9}{9} = 5$, and the average value of the second and third digits is $\frac{0+1+2+3+4+5+6+7+8+9}{10} = 4.5$ each. There are $9 \cdot 10 \cdot 10 = 900$ three digit numbers. For the second case, the average value of the first digit is $\frac{1+2+3+4+5+6+8+9}{8} = \frac{19}{4}$, and the average value of the second and third digits is $\frac{0+1+2+3+4+5+6+8+9}{9} = \frac{38}{9}$ each. There are $8 \cdot 9 \cdot 9 = 648$ three digit numbers with no digit that is a 7. Thus the answer is

$$900 \cdot (5 + 4.5 + 4.5) - 648 \cdot \left(\frac{19}{4} + \frac{38}{9} + \frac{38}{9}\right) = \boxed{4050}$$

21. [**11**] A non-degenerate triangle has sides of length $\frac{1}{x}$, 1, and x. Compute the set of possible values for its area.

Solution

Let θ be the angle opposite the side of length 1. Then the area of the triangle is $\frac{1}{2} \cdot x \cdot \frac{1}{x} \cdot \sin\theta = \frac{1}{2}\sin\theta$. So, we just need to find the range of $\sin\theta$. By the Law of Cosines,

$$1^2 = x^2 + \left(\frac{1}{x}\right)^2 - 2 \cdot x \cdot \frac{1}{x} \cdot \cos\theta$$

So, using AM-GM,

$$\cos\theta = \frac{x^2 + \frac{1}{x^2}}{2} - \frac{1}{2} \geq \sqrt{x^2 \cdot \frac{1}{x^2}} - \frac{1}{2} = \frac{1}{2}$$

Since $\cos\theta \geq \frac{1}{2}$, we must have $\theta \leq 60°$. Additionally, $\theta > 0°$. So, $0 < \sin\theta \leq \frac{\sqrt{3}}{2}$. So, the answer should be $\boxed{\left(0, \frac{\sqrt{3}}{4}\right]}$.

We still must check that all values in this interval are attainable. To check this, we find the range of possible values for x. Without loss of generality, $x \geq 1$. Then by the triangle inequality,

$$1 + \frac{1}{x} > x \Leftrightarrow 1 > x - \frac{1}{x} \Leftrightarrow 1 > x^2 + \frac{1}{x^2} - 2 \Leftrightarrow x^2 + \frac{1}{x^2} < 3$$

Looking at the equality case, $x^2 + \frac{1}{x^2} = 3$, gives $\cos\theta = 1$ and $\theta = 0°$. So, any value of θ with $0° < \theta \leq 60°$ is attainable.

22. [**12**] Hanna placed 2018 candies in a circle and labeled them $1, 2, 3, \dots 2018$ in order. Kimi eats candy 1, skips candy 2, eats candy 3, and continues going around the circle eating every other candy until he has eaten all of them. Compute the label of the last candy Kimi eats.

Solution

If there were 2^n candies labeled 1 to 2^n and Kimi started at candy 1, then the last one he eats would be candy 2^n. So, after Kimi eats $2018 - 1024 = 994$ candies, there will be 1024 left and the last one remaining will be the one he passed after eating his 994^{th} candy. The first 994 candies he eats are $1, 3, 5, \ldots, 1987$, so the answer is $\boxed{1988}$.

23. [**12**] In convex quadrilateral $ABCD$, the diagonals intersect at point P. If $\tan\angle BAC = 1$, $\tan\angle DAC = 2$, $\tan\angle DCA = 3$, and $\tan\angle BCA = 4$, compute $\tan\angle BPC$.

Solution

We use coordinates. Let $A = (0,0)$, $C = (1,0)$, $B = (x_b, y_b)$, $D = (x_d, -y_d)$, with $x_b, y_b, x_d, y_d > 0$ and $x_b, x_d < 1$. Dropping perpendiculars from B and D to the x-axis and using right triangle trigonometry, we get:

$$\frac{y_b}{x_b} = 1,\ \frac{y_d}{x_d} = 2,\ \frac{y_d}{1 - x_d} = 3,\ \frac{y_b}{1 - x_b} = 4$$

Cross-multiplying gives a system of linear equations that is solved relatively easily for $B = (\frac{4}{5}, \frac{4}{5})$ and $D = (\frac{3}{5}, -\frac{6}{5})$. The answer is the slope of line BPD, which is:

$$\frac{\frac{4}{5} + \frac{6}{5}}{\frac{4}{5} - \frac{3}{5}} = \boxed{10}$$

24. [**12**] $\sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{30}$ is a root of a polynomial with integer coefficients and degree n. Compute the minimum possible value of n.

Solution

Let $x = \sqrt{1} + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{30}$. Then if we repeatedly rearrange the terms in this equation and square both sides until there are no square roots, we will have a polynomial with x as a root. Note that we only care about the degree of this polynomial, which depends only on the number of times we square the equation. To eliminate all of the square roots, we will repeat the following steps 10 times, once for each prime $p \leq 30$:

(a) Simplify all square roots, so that each number under a square root is squarefree (not divisible by a perfect square).

(b) Move all terms containing a square root with a factor of p to one side and all other terms to other side.

(c) Square both sides of the equation.

When we do these steps, we will eliminate factors of p from under all square roots. So, after 10 repetitions, we will have an integer polynomial with x as a root. The degree will be $2^{10} = \boxed{1024}$, so this should be the answer.

Now we must prove that any integer polynomial with x as a root will have degree at least 1024. To do this we will use the idea of conjugates. We define the conjugate of an expression with respect to a prime p to be the same expression, but with each factor of $\sqrt{p}$ in each term replaced with a factor of $-\sqrt{p}$.

Let $\overline{a}$ be the conjugate of a with respect to p. Then note that we have the following properties, just as with normal conjugates:

$$\overline{a+b} = \overline{a} + \overline{b},\ \overline{ab} = (\overline{a})(\overline{b}),\ \overline{n} = n$$

where n is an integer. Since an integer polynomial only adds and multiplies the input with itself and integers, this implies that $P(\overline{a}) = \overline{P(a)}$ for an integer polynomial P. This also means that if a is a root of P then $\overline{a}$ is a root of P, since $P(\overline{a}) = \overline{P(a)} = \overline{0} = 0$.

In the original problem, let P be a polynomial with x as a root. Then $\overline{x}$ is also a root, where the conjugate is with respect to any prime. Taking the conjugate with respect to any subset of the primes less than or equal to 30 will give us a new number that must be a root of P. Since there are $2^{10} = 1024$ subsets of the primes less than or equal to 30, P must have at least 1024 roots, so its degree must be at least 1024.