# 2018 Fall Stuyvesant Team Contest Solutions

1. [**5**] Stan flips an unfair coin. Given this information, what is the probability that it comes up heads?

   *Solution*

   The given information does not distinguish between heads and tails, so by symmetry the answer is $\boxed{\frac{1}{2}}$.

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2. [**5**] Find the least positive integer $n$ such that the second digit of $11^n$ is not $n$.

   *Solution*

   $$11^1 = 11, 11^2 = 121, 11^3 = 1331, 11^4 = 14641, 11^5 = 161051$$

   Therefore, $n = \boxed{5}$.

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3. [**5**] The number 2018 has the property that its first digit and last digit sum to a number which is represented by the reverse of the other digits; that is, $2 + 8 = 10$. What is the next positive integer with this property?

   *Solution*

   If the first digit is 2 and the second digit is 0, then the first and last digit must sum to a multiple of 10, so 2018 is the only possibility. If the second digit is 1 instead of 0, since the first digit is 2, the sum of the first and last digits is at least 11. Therefore, the answer is $\boxed{2119}$.

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4. [**6**] Compute the number of proper divisors of 1023.

   **Note:** A positive integer $m$ is a proper divisor of a positive integer $n$ if $\frac{n}{m}$ is an integer greater than 1.

   *Solution*

   $$1023 = 2^{10} - 1 = (2^5 - 1)(2^5 + 1) = 31 \cdot 33 = 3^1 \cdot 11^1 \cdot 31^1$$

   To choose a positive divisor of 1023, we must choose whether it is divisble by each of 3, 11, and 31. This gives $2 \cdot 2 \cdot 2 = 8$ choices. However, we have overcounted 1023 as it is not a proper divisor, so the number of proper divisors is $8 - 1 = \boxed{7}$.

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5. [**6**] Find the least positive integer $n$ such that a regular $n$-gon has interior angles of greater than $179°$.

   *Solution*

   Since the exterior angles of any polygon add up to $360°$, each interior angle has measure

   $$180° - \frac{360°}{n} < 179°$$

   We can rearrange this to $n > 360$, so the minimum integer value is $n = \boxed{361}$.

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6. [**6**] A Stuyvesant student sleeps at 11:59 pm and wakes up at 12:01 am. How many times during a Stuyvesant student's day (while they are awake) do the minute hand and hour hand of a clock line up?

   *Solution*

   The minute hand moves around the clock in 1 hour and the hour hand moves around the clock in 12 hours. Thus every 12 hours the minute hand will pass the hour hand 11 times. In 24 hours this will occur 22 times. However we subtract one occurrence for midnight, leaving $\boxed{21}$.

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7. [**7**] If $x + y = 7$ and $\frac{1}{x} + \frac{1}{y} = 0.7$, compute the greater of the two values $x$ and $y$.

*Solution*

$\frac{1}{x} + \frac{1}{y} = \frac{x+y}{xy} = \frac{7}{xy} = 0.7 \Rightarrow xy = 10$. Since $y = 7 - x \Rightarrow x(7 - x) = 10 \Rightarrow (x - 2)(x - 5) = 0$. Thus $x = 2$ and $y = 5$ or $x = 5$ and $y = 2$. Therefore, the greater value of $x$ and $y$ is $\boxed{5}$.

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8. [**7**] Compute the sum of all $x$ satisfying $4^x + 128 = 3 \cdot 2^{x+3}$.

*Solution*

We can write this as a quadratic in $2^x$: $(2^x)^2 - 24 \cdot 2^x + 128 = 0$. Since $128 \neq (\frac{24}{2})^2$, this quadratic has 2 distinct roots. By Vieta's formulas, the product of these roots is $128 = 2^7$. So the sum of the two possible values of $x$ is $\boxed{7}$.

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9. [**7**] How many ways can Kimi get from the second floor to the fourth floor using any combination of the three staircases and the two relevant escalators? (He may only go up and across floors)

*Solution*

There are 4 ways (3 stairs and the 2-3 escalator) for Kimi to go from the second floor to the third floor and 3 ways to go from the third floor to the fourth floor. Therefore there are $3 \cdot 4 = 12$ ways to go to the fourth floor stopping at third floor. There is one way (escalator) to go directly from second floor to the fourth floor. So the answer is $12 + 1 = \boxed{13}$.

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10. [**8**] Let $s(n)$ denote the sum of the digits of $n$, and let $f(n) = 11s(n) - n$. For how many two digit numbers $n$ is $f(n)$ also a two digit number?

*Solution*

Let $n = 10a + b$ for digits $a, b$. Then $s(n) = a + b$ so $f(n) = 11(a + b) - (10a + b) = a + 10b$. For $n$ to be a two digit number, we need $a > 0$. For $f(n)$ to be a two digit number, we need $b > 0$. This gives 9 choices for each of $a$ and $b$, and so we have $9 \cdot 9 = \boxed{81}$ possible values for $n$.

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11. [**8**] Compute the number of integers $n$ such that $2 \leq n \leq 2018$ and $\binom{n}{2}$ is relatively prime to 2018.

*Solution*

Note that the prime factorization of 2018 is $2 \cdot 1009$.

We have $\binom{n}{2} = \frac{n(n-1)}{2}$. This is divisible by 2 when $n(n-1)$ is divisible by 4, which happens exactly when $n$ is 1 more than a multiple of 4 or $n$ is a multiple of 4. So, we must have that $n$ is 2 or 3 more than a multiple of 4. For $2 \leq n \leq 2018$, there are 1009 such numbers. Additionally, $\frac{n(n-1)}{2}$ is divisible by 1009 when $n$ or $n - 1$ is a multiple of 1009. For $2 \leq n \leq 2018$, this happens with $n = 1009, 1010, 2018$. We had previously included 1010 and 2018, so the answer is $1009 - 2 = \boxed{1007}$.

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12. [**8**] If a regular 2018-gon $A_1A_2 \cdots A_{2018}$ has area 2018, compute $[\triangle A_1A_2A_{1010}]$.

*Solution*

Let $O$ be the center of the polygon. Note that $A_1$ and $A_{1010}$ are opposite, so $O$ lies on the diagonal $A_1A_{1010}$. If we consider $A_1A_{1010}$ to be the base of $\triangle A_1A_2A_{1010}$ and $A_1O$ to be the base of $\triangle A_1A_2O$, we notice that they have the same height (namely, perpendicular from $A_2$ to $A_1A_{1010}$). Also note that $\triangle A_1A_2O$ is exactly $\frac{1}{2018}$ of the whole polygon, so it has area 1. However, one of the bases ($A_1A_{1010}$) is twice the other ($A_1O$), so the area of $\triangle A_1A_2A_{1010}$ is twice the area of $\triangle A_1A_2O$, which is $1 \times 2 = \boxed{2}$.

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13. [**9**] Positive reals $a$ and $b$ satisfy

$$\sqrt{ab} = \sqrt{a} + \sqrt{b} + \sqrt{a+b}$$

Compute $\sqrt{ab} - 2\sqrt{a+b}$.

*Solution*

$$\begin{aligned} & \sqrt{ab} = \sqrt{a} + \sqrt{b} + \sqrt{a+b} \\ \Rightarrow \quad & \sqrt{ab} - \sqrt{a+b} = \sqrt{a} + \sqrt{b} \\ \Rightarrow \quad & (\sqrt{ab} - \sqrt{a+b})^2 = (\sqrt{a} + \sqrt{b})^2 \\ \Rightarrow \quad & ab - 2\sqrt{ab}\sqrt{a+b} + a + b = a + 2\sqrt{ab} + b \\ \Rightarrow \quad & ab - 2\sqrt{ab}\sqrt{a+b} = 2\sqrt{ab} \\ \Rightarrow \quad & \sqrt{ab} - 2\sqrt{a+b} = \boxed{2} \end{aligned}$$

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14. [**9**] A random positive integer less than or equal to 900 is chosen. What is the probability that the quotient when divided by 30 is greater than the remainder when divided by 30?

*Solution*

We will first find the probability if the integer is chosen from 0 through 899, inclusive. Note that for any ordered pair of non-negative integers $(a, b)$, where $a$ and $b$ are less than 30, there is exactly one integer in the interval $[0, 899]$ that has quotient $a$ and remainder $b$ when divided by 30. Thus, since there are an equal number of pairs with $a > b$ and pairs with $b > a$, the probability that $a > b$ is $\frac{1-p}{2}$, where $p$ is the probability that $a = b$. It can be easily proved that $p = \frac{1}{30}$, so $\frac{1-p}{2} = \frac{1-\frac{1}{30}}{2} = \frac{29}{60}$. However, since the quotient when 900 is divided by 30 is clearly greater than its remainder, and this is not the case with 0, we must add an extra $\frac{1}{900}$. Thus, our answer is $\frac{29}{60} + \frac{1}{900} = \boxed{\frac{109}{225}}$

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15. [**9**] The polynomial $x^3 + 4x^2 + bx + c$ has roots $r$, $s$, and $t$. If $r^2 + s^2 + t^2 = 16$, compute $bc$.

*Solution* By Vieta's, $r + s + t = -4$ and $rs + st + tr = b$. We have:

$$r^2 + s^2 + t^2 + 2(rs + st + tr) = (r + s + t)^2$$

So, $16 + 2b = (-4)^2$, which implies $b = 0$. Thus $bc = \boxed{0}$.

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16. [**10**] How many three digit multiples of 9 have two of the digits summing to the third?

*Solution*

Suppose that the digits are $a, b, c$ such that $a + b = c$. Then since the number is a multiple of 9, $a + b + c = 2c$ is a multiple of 9. So, $c$ is a multiple of 9. So, we must have that our number has at least one digit that is a 9, and the other two digits must sum to 9. To count the number of such numbers, first assume that the the digits have no 0's and exactly one 9. Then we have 3 choices for which digit is the 9, and 8 choices for the pair of the other digits, from $(1, 8)$ to $(8, 1)$. This gives 24 such numbers. Additionally, if we have multiple 9's or we have a 0, the only two such numbers are 990 and 909. So the answer is $24 + 2 = \boxed{26}$.

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17. [**10**] In $\triangle ABC$, $AB = 13$, $BC = 14$ and $AC = 15$. Let $M_1$ and $M_2$ be the trisection points of $BC$ and consider the circles with diameters $AM_1$ and $AM_2$. Suppose the circles intersect again at a point $X \neq A$. Compute $AX$.

*Solution*

We have $\angle AXM_1 = \angle AXM_2 = 90°$. So $X$ lies on line $M_1M_2$, which is line $BC$. So, $X$ is the foot of the perpendicular from $A$ to $BC$. So, $AX = \boxed{12}$ (we can find this by using areas).

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18. [**10**] Suppose that $P(x)$ is a cubic polynomial satisfying $P(1) = 4$, $P(2) = 9$, and $P(3) = 16$. Compute $P(0) + P(4)$.

*Solution*

Let $Q(x) = P(x) - (x+1)^2$. Note that $Q(x)$ is a cubic with roots 1, 2, and 3.

So $Q(x) = a(x-1)(x-2)(x-3)$, for some $a$.

Then $P(0) + P(4) = Q(0) + (0+1)^2 + Q(4) + (4+1)^2 = -6a + 1 + 6a + 25 = \boxed{26}$.

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19. [**11**] Two chords of a circle have length 12 and 13. They intersect, forming an angle of 30°. If one of the chords bisects the other, compute the radius of the circle.

*Solution*

If the bisected chord has length 13, then it is split into two equal pieces of length 6.5. Then by power of a point on the intersection of the chords, we have two lengths summing to 12 and multiplying to $6.5^2$. But this is not possible because the maximum product we could have is $6^2$, which is less than $6.5^2$. So the chord of length 12 is bisected. Then by power of a point, we see that the chord of length 13 is split into pieces of length 4 and 9.

Let $O$ be the center of the circle, let $P$ be the intersection point of the chords, and let $Q$ be the midpoint of the chord of length 13. Then $PQ = 2.5$. Since $O$ is the center of the circle, $OQ$ is perpendicular to the chord of length 13. In addition, $OP$ is perpendicular to the chord of length 12. Using the givens, $\angle OPQ = 60°$. So $OP = 5$ by $30 - 60 - 90$ triangles.

Let $R$ be an endpoint of the chord of length 12. Then by the Pythagorean Theorem on $\triangle OPR$, the radius ($OR$) is $\sqrt{5^2 + 6^2} = \boxed{\sqrt{61}}$.

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20. [**11**] Let $A$ be an arithmetic sequence. Compute the smallest positive common difference $d$ such that 6 consecutive terms of $A$ are primes.

*Solution*

If the common difference $d$ is an odd integer, then any six consecutive terms in $A$ will contain three even numbers, all of which cannot be prime, so $d$ must be even. If $d$ is not a multiple of three, then of any three consecutive terms, at least one is divisible by 3, which means two numbers in this sequence will be a multiple of 3, which is impossible, so 3 must divide $d$. If $d$ is not divisible by 5, then any consecutive five terms contains a multiple of 5, so the only way for $A$ to contain six consecutive primes is if one of the last five of these six primes is the prime 5. However, if this is the case, since $d$ is divisible by 6 (and thus $d > 5$), the first of these six terms would not be a positive integer, which is not possible. Thus, 5 divides $d$. Since 2, 3, and 5 all divide $d$, we have 30 divides $d$, so $d$ is at least 30. Now, note that $7, 37, 67, 97, 127, 157$ are six prime numbers that form an increasing arithmetic progression with common difference 30, so the answer is $\boxed{30}$.

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21. [**11**] Compute the sum of all integers $n$ for which

$$\frac{n^4 + n^2 + 400}{n^2 + n + 1}$$

is also an integer.

*Solution*

We have:

$$\frac{n^4+n^2+400}{n^2+n+1} = \frac{n^4+n^2+1+399}{n^2+n+1} = \frac{(n^2+n+1)(n^2-n+1)+399}{n^2+n+1} = n^2-n+1+\frac{399}{n^2+n+1}$$

So, the given condition is equivalent to $n^2+n+1$ being a divisor of 399. Let $f(n) = n^2+n+1$. Notice that $f(n) = f(-1-n)$. So, we can pair the desired values of $n$ into pairs summing to $-1$. So, we count the number of nonnegative values of $n$ for which $f(n)$ divides $399 = 3*7*19$. We obtain the following values: $f(0) = 1$, $f(1) = 3$, $f(2) = 7$, $f(4) = 19$, $f(7) = 57 = 3 \cdot 19$, $f(11) = 133 = 7 \cdot 19$. So the answer is $\boxed{-6}$.

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22. [**12**] How many ways can the squares of a 2 by 3 board be filled with elements of $\{-1, 0, 1\}$ if adjacent squares cannot sum to 0?

*Solution*

We will fill the elements column by column, through the three columns. The top element of the first column has 3 options and the bottom element then has 2 options. Now, for each possible (ordered) pair of elements chosen in the first column, there are three possible pairs of elements that could go in the second column. So we have 3 options for the second column. Similarly we have 3 options for the third column. So the answer is $3 \cdot 2 \cdot 3 \cdot 3 = \boxed{54}$.

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23. [**12**] Positive real numbers $a, b, c, d$ satisfy

$$a^2 - \sqrt{3}ab + b^2 = c^2 + \sqrt{3}cd + d^2 = 81$$

Find the maximum value of $ac + bd$.

*Solution*

Consider a triangle with side lengths of $a$ and $b$ with a $150°$ angle between them. By the Law of Cosines, the third side of this triangle has length 9. Similarly we construct a triangle with side lengths of $c$, $d$, and 9, with a $30°$ angle opposite the side of length 9. Putting these two triangles together, we obtain a quadrilateral with sides of length $a$, $b$, $c$, and $d$ (in that order). Let $e$ be the distance between the two vertices not on the diagonal of length 9. Notice that this quadrilateral is cyclic as $150°$ and $30°$ are supplementary. So $ac + bd = 9e$ by Ptolemy's Theorem. Also notice that the diameter of this circle is $\frac{9}{\sin 30°} = 18$ by the Extended Law of Sines. Since $e$ is a chord of this circle, $e \leq 18$ ($e = 18$ is easily attainable). So $ac + bd \leq 9 \cdot 18 = \boxed{162}$.

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24. [**12**] How many ordered triples of integers $(a, b, c)$ satisfy $1 \leq a, b, c \leq 30$ and $(1+2a)(1+3b)(1+5c)-1$ is divisible by 30?

*Solution*

Note that the remainder of $(1+2a)(1+3b)(1+5c)-1$ upon division by 30 does not change if $a$ is increased or decreased by 15, or $b$ is increased or decreased by 10, or $c$ is increased or decreased by 6. So, we assume that $1 \leq a \leq 15$, $1 \leq b \leq 10$, and $1 \leq c \leq 6$. We will multiply our answer by $2 \cdot 3 \cdot 5 = 30$ at the end.

Let $x = 1+2a$, $y = 1+3b$, $z = 1+5c$. Then the information we have is equivalent to choosing residues $x, y, z \pmod{30}$ such that $x \equiv 1 \pmod 2$, $y \equiv 1 \pmod 3$, $z \equiv 1 \pmod 5$ and $xyz \equiv 1 \pmod{30}$. By the Chinese Remainder Theorem, we just need to choose remainders for $x$, $y$, and $z$ upon division by 2, 3, and 5. Let $p$ be one of these primes. Then $\pmod p$, one of the variables is fixed at 1 and the product of all three is also 1. So, the product of the other 2 is also 1. Since multiplicative inverses exist for nonzero elements $\pmod p$, we have $p-1$ choices for one of the other two variables and the other variable is forced. Thus we have $(2-1)(3-1)(5-1) = 8$ choices for $x, y, z$. So the answer is $8 \cdot 30 = \boxed{240}$.

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