2017 Stuyvesant Team Contest Solutions

1. [**5**] Compute:
$$2 \cdot 4 \cdot 6 - 2 \cdot 4 - 4 \cdot 6 - 6 \cdot 2 + 2 + 4 + 6.$$

   Solution

   $2 \cdot 4 \cdot 6 - 2 \cdot 4 - 4 \cdot 6 - 6 \cdot 2 + 2 + 4 + 6 = (2-1)(4-1)(6-1) + 1 = 1 \cdot 3 \cdot 5 + 1 = 15 + 1 = \boxed{16}$

2. [**5**] Compute the least positive integer k such that k is not a multiple of 3 and $10k + 3$ is not prime.

   Solution

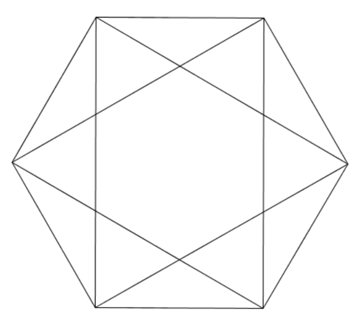
   Testing values of k that are not multiples of 3, we find that $\boxed{13}$ is the first value that works, since $13 \cdot 10 + 3 = 133 = 7 \cdot 19$.

3. [**5**] If $a = 8$, $b = 15$, and $c = 17$, compute:
$$\frac{\frac{a+|a-b|+b}{2} + \left|\frac{a+|a-b|+b}{2} - c\right| + c}{2}$$

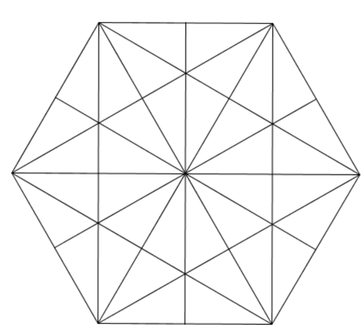
   Solution

   Note that $\frac{a+|a-b|+b}{2} = \max(a, b)$. Applying this twice, we get that the expression is equal to $\max(\max(a, b), c)$. Substituting the values of a, b, and c, we get $\boxed{17}$.

4. [**6**] In the diagram below, the vertices of the smaller regular hexagon are intersections of the diagonals of the larger regular hexagon. Compute the ratio of the area of the smaller regular hexagon to the larger.



   Solution



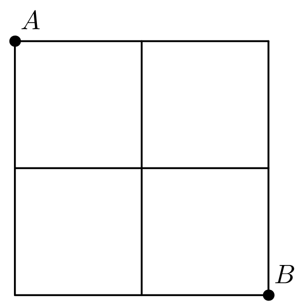
Construct 36 congruent $30-60-90$ triangles as shown. Let the area of each be K. Then the areas of the hexagons are $12K$ and $36K$, so the answer is $\frac{12K}{36K} = \boxed{\frac{1}{3}}$.

5. [**6**] A random number generator outputs the numbers $6, 3$, and 2 with probability $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{6}$, respectively. Compute the average value of the output.

   Solution

   Expected value is the sum of possible values, each weighted (multiplied) by their probability. Thus, the answer is $6 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{6} = \boxed{\frac{13}{3}}$.

6. [**6**] Compute the number of paths from A to B along the lattice grid that pass through each of the 9 intersection points at most once.



   Solution

   We do casework on the number of moves in the 'wrong' direction (moves up or to the left). If there are no moves in the wrong direction, there are $\binom{4}{2} = 6$ paths. If there is exactly one move in the wrong direction, we trace out possible paths and find that there are 4. If there are exactly two moves in the wrong direction, we trace out possible paths and find that there are 2. It is not possible to have more than two moves in the wrong direction, so the answer is $6 + 4 + 2 = \boxed{12}$.

7. [**7**] In equilateral triangle ABC with side length 8, D is the foot of the altitude from A to BC and E is the foot of the altitude from D to AC. Compute $[CDE]$.

   Note: $[\cdots]$ denotes the area of the polygon in the brackets.

   Solution

   Since ABC is equilateral, D is the midpoint of $\overline{BC}$, meaning $DC = 4$. We note that $\angle DAC = 30°$, so $\triangle ADC$ is a $30-60-90$ triangle. Since $\overline{DE}$ is an altitude, $\angle CED = 90°$, so CDA is a $30-60-90$ triangle. We thus compute $EC = 2$ and $DE = 2\sqrt{3}$, to find:

$$[ADE] = \frac{1}{2} \cdot EC \cdot DE = \frac{1}{2} \cdot 2 \cdot 2\sqrt{3} = \boxed{2\sqrt{3}}.$$

8. [**7**] Akash, Matthew, and Milan are playing rock-paper-scissors. If each of them randomly chooses their play, compute the probability that Akash beats Matthew, Matthew beats Milan, and Milan beats Akash.

   Solution

The total number of possible plays is $3 \cdot 3 \cdot 3 = 27$, because Akash, Matthew, and Milan each have 3 choices. Additionally, once Milan picks his play, there is only one set of plays for Akash and Matthew to satisfy the condition. Thus, there are 3 sets of plays that satisfy the condition, and the answer is $\frac{3}{27} = \boxed{\frac{1}{9}}$.

9. [**7**] Compute

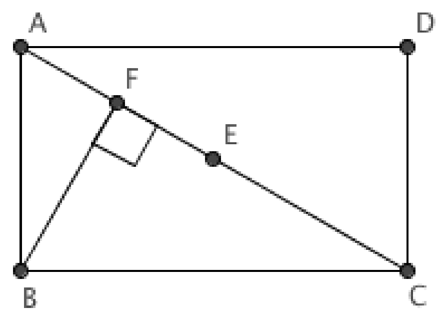
$$(\log_2 2017)(\log_3 2016)(\log_4 2015) \cdots (\log_{2015} 4)(\log_{2016} 3)(\log_{2017} 2).$$

Solution

By change of base formula, $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$. So, $(\log_b a)(\log_a b) = 1$. Applying this identity repeatedly, all of the terms in the product cancel, giving an answer of $\boxed{1}$.

10. [**8**] In rectangle $ABCD$, E is the center and F is the midpoint of segment AE. If $BF \perp AC$ and $AF = 1$, compute the $[ABCD]$.

Solution



First, compute $AC = 2AE = 4AF = 4$ and $FC = AC - AF = 4 - 1 = 3$. Now, note that we have similar triangles $\triangle AFB \sim \triangle ABC \sim \triangle BFC$ by AA similarity. So,

$$\frac{AF}{AB} = \frac{AB}{AC} \text{ and } \frac{FC}{BC} = \frac{BC}{AC}$$

This means that

$$(AB)^2 = (AF)(AC) = 4 \text{ and } (BC)^2 = (FC)(AC) = 12$$

Then the area of the rectangle is $(AB)(BC) = \sqrt{4} \cdot \sqrt{12} = \boxed{4\sqrt{3}}$.

11. [**8**] Compute the maximum number of regions one circle and three lines can divide the plane into. (A region can have finite or infinite area).

Solution

The maximum number of regions that they can split the plane into is 7 (this can be seen by drawing). Now we are going to draw a circle and try to maximize the number of regions formed. Note that the circle intersects each line at most twice, leading to 6 total points of intersections between the circle and a line. Note that for each of the 6 pairs of consecutive intersection points, the arc between them creates at most 1 new region. Thus our answer is $6 + 7 = \boxed{13}$.

12. [**8**] Let F_n denote the Fibonacci sequence, defined as $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Let s_1 be the sum of the positive integer solutions a to the equation $F_a = a$. Let s_2 be the sum of the positive integer solutions b to the equation $F_b = b^2$. Compute $F_{s_1} + F_{s_2}$.

    Solution

    We list out the first few terms of the Fibonacci sequence and find that $F_1 = 1$, $F_5 = 5$, $F_1 = 1^2$, and $F_{12} = 12^2$. Note that for $a > 5$, $F_a > a$ and for $b > 12$, $F_b > b^2$. So, these are the only solutions. Thus $s_1 = 1 + 5 = 6$ and $s_2 = 1 + 12 = 13$. The answer is $F_6 + F_{13} = 8 + 233 = \boxed{241}$.

13. [**9**] Compute:
$$100^2 - 2 \cdot 99^2 + 3 \cdot 98^2 - 4 \cdot 97^2 + \cdots + 99 \cdot 2^2 - 100 \cdot 1^2.$$

    Solution

$$\begin{aligned}
&100^2 - 2 \cdot 99^2 + 3 \cdot 98^2 - 4 \cdot 97^2 + \cdots + 99 \cdot 2^2 - 100 \cdot 1^2 \\
=&100^2 - 99^2 - 99^2 + 98^2 + 2 \cdot 98^2 - 2 \cdot 97^2 - 2 \cdot 97^2 + 2 \cdot 96^2 + 3 \cdot 96^2 - 3 \cdot 95^2 \\
&- \cdots + 49 \cdot 4^2 - 49 \cdot 3^2 - 49 \cdot 3^2 + 49 \cdot 2^2 + 50 \cdot 2^2 - 50 \cdot 1^2 - 50 \cdot 1^2 \\
=&(100^2 - 99^2) - (99^2 - 98^2) + 2 \cdot (98^2 - 97^2) - 2 \cdot (97^2 - 96^2) + 3 \cdot (96^2 - 95^2) \\
&- \cdots + 49 \cdot (4^2 - 3^2) - 49 \cdot (3^2 - 2^2) + 50 \cdot (2^2 - 1^2) - 50 \\
=&(100 + 99) - (99 + 98) + 2 \cdot (98 + 97) - 2 \cdot (97 + 96) + 3 \cdot (96 + 95) \\
&- \cdots + 49 \cdot (4 + 3) - 49 \cdot (3 + 2) + 50 \cdot (2 + 1) - 50 \\
=&(100 - 98) + 2 \cdot (98 - 96) + 3 \cdot (96 - 94) + \cdots + 49 \cdot (4 - 2) + 50 \cdot 2 \\
=&2 + 2 \cdot 2 + 3 \cdot 2 + \cdots + 49 \cdot 2 + 50 \cdot 2 = 2 \cdot (1 + 2 + 3 + \cdots + 49 + 50) \\
=&2 \cdot \frac{50 \cdot 51}{2} = 50 \cdot 51 = \boxed{2550}
\end{aligned}$$

14. [**9**] Compute the sum of all positive integers a such that $2^a + a^2$ is either a power of 2 or a perfect square.

    Solution

    Checking, small cases for a, we find the solutions 2, 4, and 6. So, the answer should be $\boxed{12}$. We will show that there are no other solutions, which is a bit more difficult.

    Let $2^a + a^2 = 2^k$ for a positive integer $k \geq a$. Testing out $a = 1, 2, 3, 4$ we find that $a = 2, 4$ give $k = 3, 5$ as solutions. If $a > 4$, $2^a > a^2$. So,
$$2^k = 2^a + a^2 < 2^a + 2^a = 2^{a+1} \leq 2^k$$
    which is a contradiction.

    Now suppose $2^a + a^2 = m^2$ for a positive integer m. Rearranging gives
$$2^a = m^2 - a^2 = (m - a)(m + a)$$
    We split the problem in two cases, a is odd and a is even.

    If a is odd, then m is odd as well. Since $\gcd(x, y) = \gcd(x, x - y)$,
$$\gcd(m - a, m + a) = \gcd(m - a, m + a - (m - a)) = \gcd(m - a, 2a).$$

Since the greatest common divisor of $m-a$ and $m+a$ must divide 2^a, it must be a power of two. Further, because a is odd, the greatest common divisor must be 1 or 2. If $a \geq 5$, then $m \geq 7$, producing factors that are relatively prime with 2^a and giving a contradiction. $a = 3$ and $a = 1$ do not work.

Otherwise, if a is even, we write $a = 2^b \cdot a_1$ and $m = 2^c \cdot m_1$, where a_1 and m_1 are odd. So,

$$2^a = 2^{2c}m_1^2 - 2^{2b}a_1^2$$

We must then have $b = c$, because otherwise the greatest power of 2 dividing the left side would not equal the greatest power of 2 dividing the right side (we are again assuming that $a > 4$ so that $2^a > a^2$). Substituting $b = c$ gives:

$$2^a = 2^{2b}(m_1^2 - a_1^2)$$

We divide by 2^{2b}:

$$2^{a-2b} = m_1^2 - a_1^2,$$

and we proceed as in the odd case up to the last step. If $a_1 = 3$, then $m_1 = 5$ gives the solution $a = 6$. $a_1 = 1$ implies $m_1 = 3$, which does not work. Thus, our only solutions are $a \in \{2, 4, 6\}$.

15. [**9**] Milan rolls a 20 sided die labeled with the numbers $1, 2, 3, ..., 20$. Matthew rolls a 17 sided die labeled $1, 2, ..., 17$. Compute the probability Matthew's roll is greater than Milan's roll.

Solution

First, if Milan rolls 18, 19 or 20 Matthew is guaranteed to lose. Now we will only consider cases where Milan rolls a 1, 2, ... 17 (with probability $\frac{17}{20}$). Then by symmetry, the probability Matthew wins is equal to the probability Milan wins. Adding the probability of a draw (both roll the same number) to these two probabilities gives 1. The probability of them drawing, both rolling a number from 1 to 17 is $\frac{17}{17^2} = \frac{1}{17}$. Let p be the probability that Matthew wins. Then we have $2p + \frac{1}{17} = 1$. So, $p = \frac{1}{2} \cdot \left(1 - \frac{1}{17}\right)$. Therefore, taking into account the $\frac{17}{20}$, we have that the answer is:

$$\frac{17}{20} \cdot \left(\frac{1}{2} \cdot \left(1 - \frac{1}{17}\right)\right) = \boxed{\frac{2}{5}}$$

16. [**10**] Compute the least positive integer N such that $N^N > 10^{20}$.

Solution

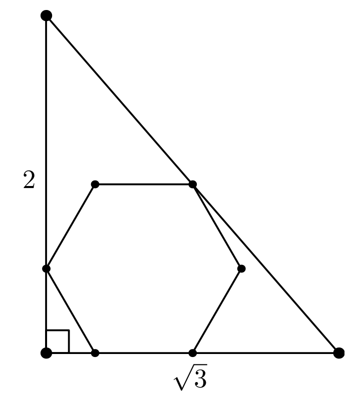
First, note that $10 < N < 20$. So, we try values of N between 10 and 20. Since $2^{10} = 1024 \approx 1000 = 10^3$, we have that

$$16^{16} = 2^{64} = (2^{10})^6 \cdot 2^4 \approx (10^3)^6 \cdot 2^4 = 16 \cdot 10^{18} < 100 \cdot 10^{18} = 10^{20}$$

So, 16 is too small of a value for N, but it is close. Thus, we expect $\boxed{17}$ to be the answer. This can be checked with the same technique:

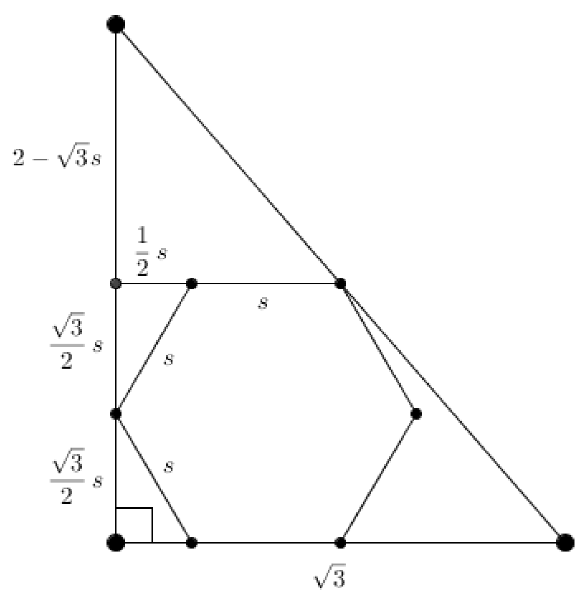
$$17^{17} > 16^{17} = 2^{68} = (2^{10})^6 \cdot 2^8 \approx (10^3)^6 \cdot 2^8 = 256 \cdot 10^{18} > 100 \cdot 10^{18} = 10^{20}$$

17. [**10**] Compute the side length of the regular hexagon inscribed in a right triangle with legs $\sqrt{3}$ and 2 as shown:



Solution

Let the side length of the hexagon be s. Extend the top side of the hexagon and use $30-60-90$ triangles to obtain the lengths shown:



Now, note that the smaller right triangle at the top is similar to the larger right triangle. So, the ratios of their sides are equal, and we have:

$$\frac{2}{\sqrt{3}} = \frac{2-\sqrt{3}s}{\frac{1}{2}s+s}$$

Cross-multiplying, we get that $3s = 2\sqrt{3} - 3s$, which implies $s = \boxed{\frac{\sqrt{3}}{3}}$.

18. [**10**] Compute the sum of all primes p satisfying:

$$p \mid 3^p + 11^p + 19^p.$$

Note: $a \mid b$ for integers a and b if a is a factor of b.

Solution

Fermat's Little Theorem states that $a^p \equiv a \mod p$ for all integers a and primes p. So, we have that $3^p \equiv 3 \mod p$, $11^p \equiv 11 \mod p$, and $19^p \equiv 19 \mod p$. This means that $p \mid 3^p - 3$, $p \mid 11^p - 11$, and $p \mid 19^p - 19$. If $p \mid a$ and $p \mid b$, then $p \mid a - b$. So,

$$p \mid (3^p + 11^p + 19^p) - (3^p - 3) - (11^p - 11) - (19^p - 19) \Rightarrow p \mid 3 + 11 + 19 \Rightarrow p \mid 33$$

The only primes p dividing 33 are 3 and 11. Both of these work because our steps were reversible. Thus the answer is $3 + 11 = \boxed{14}$.

19. [**11**] Let $P(x) = x^{2017} - 2017x - 2017$. Let $r_1, r_2, r_3, ..., r_{2017}$ be the zeros of P. If

$$T = r_1^{2017} + r_2^{2017} + r_3^{2017} + \cdots + r_{2017}^{2017},$$

compute the remainder when T is divided by 1000.

Solution

Note that by the definition of a root, $r_i^{2017} - 2017r_i - 2017 = 0$. Rearranging this, we get $r_i^{2017} = 2017r_i + 2017$. Thus,

$$\sum_{i=1}^{2017} r_i^{2017} = 2017 \sum_{i=1}^{2017} r_i + \sum_{i=1}^{2017} 2017$$

However, by Vieta's formulas,

$$\sum_{i=1}^{2017} r_i = 0$$

So, our sum reduces to

$$\sum_{i=1}^{2017} 2017 = 2017 \cdot 2017 = 4068289$$

Taking the remainder when dividing this by 1000, and we get $\boxed{289}$.

20. [**11**] In $\triangle ABC$, $AC = 20$, $BC = 17$. If $\cos^2 A + \cos^2 B + \sin^2 C = 2$, compute $[ABC]$.

Solution

Note that for any θ, $\cos^2\theta = 1 - \sin^2\theta$. We write our equation entirely in terms of sines:

$$\sin^2 C = 2 - (\cos^2 A + \cos^2 B) = 2 - ((1 - \sin^2 A) + (1 - \sin^2 B)) = \sin^2 A + \sin^2 B.$$

By the law of sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k,$$

for some constant k. We can then write $a = k\sin A$, $b = k\sin B$, $c = k\sin C$. Multiplying the equation above by k^2, we find:

$$(k\sin C)^2 = (k\sin A)^2 + (k\sin B)^2.$$

Finally, we substitute a, b, c, giving:

$$c^2 = a^2 + b^2$$

Thus, ABC is a right triangle with a right angle at C. Then the area is:

$$\frac{1}{2}AC \cdot BC = \boxed{170}.$$

21. [**11**] Compute the least positive integer n such that:

$$\left\lfloor \frac{n}{4} \right\rfloor + \left\lfloor \frac{n}{4^2} \right\rfloor + \left\lfloor \frac{n}{4^3} \right\rfloor + \left\lfloor \frac{n}{4^4} \right\rfloor + \cdots = 1000.$$

Note: $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Solution

First note that $\lfloor \frac{n}{4} \rfloor + \lfloor \frac{n}{4^2} \rfloor + \lfloor \frac{n}{4^3} \rfloor + \lfloor \frac{n}{4^4} \rfloor + \cdots \approx \frac{n}{4} + \frac{n}{4^2} + \frac{n}{4^3} + \frac{n}{4^4} + \cdots = \frac{\frac{n}{4}}{1-\frac{1}{4}} = \frac{n}{3}$, by the infinite geometric series formula. Since $\frac{3000}{3} = 1000$, $n = 3000$ is a good place to start. Plugging in $n = 3000$ into our expression we get $\lfloor \frac{3000}{4} \rfloor + \lfloor \frac{3000}{4^2} \rfloor + \lfloor \frac{3000}{4^3} \rfloor + \lfloor \frac{3000}{4^4} \rfloor + \cdots = 750+187+46+11+2+0+0+\cdots = 996$, so we are pretty close, but too low. Going on to the next multiple of 4, we see that $n = 3004$ only increments the sum by 1. However, plugging in $n = 3008$ gets the sum to exactly 1000, so the answer is $\boxed{3008}$.

22. [**12**] In Mr. Cocoros's 5^{th} period calculus class, only 5 students are present. Unfortunately, all 5 have fallen asleep. To wake them up, Mr. Cocoros plans to throw pieces of chalk at them. However, due to his bad aim, Mr. Cocoros misses his target half of the time and hits a random other student. Any student hit by chalk will wake up. If Mr. Cocoros has 5 pieces of chalk to throw and aims them optimally, compute the probability that he will wake up the entire class.

Solution

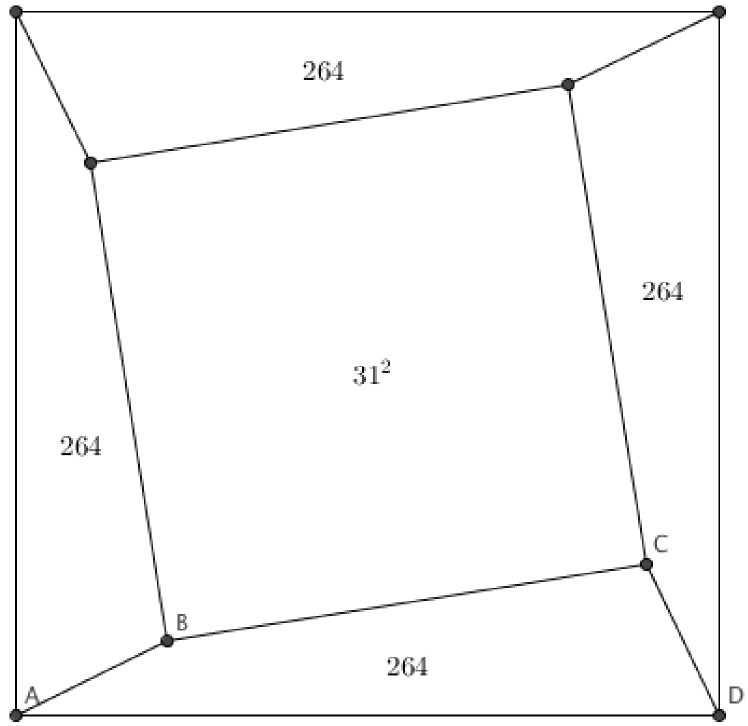
We calculate the probability that Mr. Cocoros wakes up a student on his k^{th} throw, assuming that he has woken up $k-1$ students already. He should aim for a student that is asleep. This gives a probability of $\frac{1}{2} + \frac{1}{2} \cdot \frac{4-(k-1)}{4}$, since if he misses there are still $4-(k-1)$ students he could hit by accident to wake one up. Multiplying this from $k = 1$ to $k = 5$, we have:

$$\prod_{k=1}^{5} \frac{1}{2} + \frac{1}{2} \cdot \frac{4-(k-1)}{4} = \prod_{k=1}^{5} \frac{1}{2} \cdot \left(1 + \frac{5-k}{4}\right) = \frac{1}{2^5} \prod_{k=1}^{5} \frac{9-k}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{2^5 \cdot 4^5} = \boxed{\frac{105}{512}}$$

23. [**12**] In quadrilateral $ABCD$, $AB = CD$ and $\angle BAD + \angle CDA = 90^\circ$. If $BC = 31$ and $[ABCD] = 264$, compute AD^2.

Solution

We construct squares on AD and BC as shown, giving 4 congruent quadrilaterals by the conditions in the problem.



AD^2 is the area of the larger square. Thus, the answer is $31^2 + 4 * 264 = \boxed{2017}$.

24. [**12**] An *Additive Magic Square* (AMS) is a 3 by 3 grid filled with distinct positive integers such that the sum of each row, column, and diagonal are the same. A *Multiplicative Magic Square* (MMS) is a 3 by 3 grid filled with distinct positive integers such that the product of each row, column, and diagonal are the same. The minimum possible sum of elements of an AMS is 45. Compute the minimum possible sum of elements of a MMS.

    Solution

    We claim that the answer is $\boxed{91}$. This can be attained with the following MMS:

2	36	3
9	6	4
12	1	18

We will now show that 91 is the minimum possible value for the sum of the elements of an MMS. Let P be the common product of each row, column, and diagonal. Let m be the value in the middle square, and let $a_1, a_2, \ldots, a_8$ be the values in the other squares. Then we claim that the $m^3 = P$. To prove the claim, notice that, by taking the product of all three rows, we get

$$a_1 a_2 \ldots a_8 m = P^3$$

Additionally, by taking the product of the diagonals, the center row, and the center column, we get

$$a_1 a_2 \ldots a_8 m^4 = P^4$$

Dividing these two equations gives the desired result.

Now, we will do casework on the number of distinct primes that divide elements of the MMS.

If there is only one prime that divides the elements of the MMS, then all of the elements must be distinct powers of that prime. The sum is minimized when the prime is 2 and when the elements are $2^0, 2^1, 2^2, \ldots, 2^8$, which gives a sum of $511 > 91$.

If there are three or more distinct primes that divide elements of the MMS, they must each divide P. Since $m^3 = P$, they must also divide m. So m must be at least $2 \cdot 3 \cdot 5 = 30$, the product of the three smallest primes. The product of the elements of the MMS is $P^3 = m^9$. Then, by AM-GM,

$$a_1 + a_2 + \cdots + a_8 + m \geq 9\sqrt[9]{a_1 \cdot a_2 \cdot \cdots \cdot a_8 \cdot m} = 9\sqrt[9]{m^9} = 9m \geq 9 \cdot 30 = 270 > 91$$

So, the only case left to consider is when exactly 2 distinct primes divide elements of the MMS. To minimize the sum, we let the 2 primes be 2 and 3. As before, both 2 an 3 must divide P, and also m. So, $m = 2^j \cdot 3^k$ for positive integers j and k. If either j or k is greater than 1, we have that m is greater than either $2^2 \cdot 3 = 12$ or $2 \cdot 3^2 = 18$. So, $m \geq 12$. Then, we can use AM-GM as before to obtain

$$a_1 + a_2 + \cdots + a_8 + m \geq 9\sqrt[9]{a_1 \cdot a_2 \cdot \cdots \cdot a_8 \cdot m} = 9\sqrt[9]{m^9} = 9m \geq 9 \cdot 12 = 108 > 91$$

Now we must check that 91 is the minimum possible sum when $m = 2^1 \cdot 3^1 = 6$. In this case, $P = m^3 = 2^3 \cdot 3^3 = 216$. Now, we claim that all of the elements of the MMS are of the form $2^j \cdot 3^k$, with $0 \leq j, k \leq 2$. Suppose that for some element a, $j \geq 3$ ($k \geq 3$ is analogous). Then consider the row, column, or diagonal that contains a and m. The product of the three entries must be equal to P, but the exponent of 2 in the product of the three elements is at least $3 + 1 = 4$. This is a contradiction. So, all of the elements are of the desired form.

Next, notice that there are only 3 possible values for both j and k. This means that there are only $3 \cdot 3 = 9$ possible values for the elements of the MMS. Since there are 9 elements total, and since they are distinct, we know that they must be some arrangement of these 9 possible values. The sum of the elements is then:

$$2^0 \cdot 3^0 + 2^1 \cdot 3^0 + 2^2 \cdot 3^0 + 2^0 \cdot 3^1 + 2^1 \cdot 3^1 + 2^2 \cdot 3^1 + 2^0 \cdot 3^2 + 2^1 \cdot 3^2 + 2^2 \cdot 3^2 = (2^0 + 2^1 + 2^2)(3^0 + 3^1 + 3^2) = 91$$

This completes the proof.