1. **[6]** Compute

$$(2+0+2+4)(2!+0!+2!-4^2)(2^2-0!+2^2+4^2).$$

Team Name: ____________________ Answer: ____________________

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2. **[6]** The answer to this problem can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Compute $2\sqrt{ab}$.

Team Name: ____________________ Answer: ____________________

. . . .

3. **[7]** Find the positive integer $n < 100$ for which the ratio of n to the sum of its divisors is maximal.

Team Name: ____________________ Answer: ____________________

. . . .

4. **[7]** Compute $\log_{32} 81 \cdot \log_6 7 \cdot \log_3 36 \cdot \log_{343} 2$.

Team Name: ____________________ Answer: ____________________

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5. **[8]** Let $\triangle ABC$ have area 256. The midpoints of $\overline{AB}$ and $\overline{AC}$ are D and E, respectively. Points F and G are on segments $\overline{AB}$ and $\overline{AC}$ respectively such that $\overline{FG}$ is parallel to $\overline{BC}$ and the lengths of $\overline{DE}$, $\overline{FG}$, and $\overline{BC}$ form a geometric sequence in some order. What is the sum of the possible areas of $\triangle AFG$?

Team Name: ____________________ Answer: ____________________

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6. **[8]** Which positive integer is four times a prime number and one less than a perfect cube?

Team Name: ____________________ Answer: ____________________

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7. **[9]** Quadrilateral $ABCD$ has $AB = 3$, $BD = 4$, and $CD = 5$. Find the maximum possible area of $ABCD$.

Team Name: ____________________ Answer: ____________________

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8. **[9]** Compute the number of positive integers n such that the decimal representations of n and $2n$ each have exactly one odd digit and one even digit.

Team Name: ____________________ Answer: ____________________

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9. **[10]** Four distinct congruent segments are drawn through the center of a square $\mathcal{S}$ such that the endpoints of each segment are on the perimeter of the square. If the segments partition $\mathcal{S}$ into eight pieces of area 18, what is the length of each of these segments?

Team Name: ____________________ Answer: ____________________

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10. **[Up to 10]** Welcome to **Proof by Democracy**, the minigame where you (pl.) get to decide the truth of some famous open problems!

Instructions: The following six statements are currently open: nobody knows whether they are true or false. Submit a string of 6 letters in which the kth letter is Y if you think statement k is true and N if you think it is false. If the proportion of teams that have the same answer as you for statement k is x_k, then you will receive $10\sqrt[3]{x_1x_2x_3x_4x_5x_6}$ points.

(a) (Fortune's conjecture) Denote the sequence of primes in increasing order by $p_1, p_2, p_3, \ldots$. For a given positive integer n, let m be the smallest integer greater than 1 such that $p_1p_2\cdots p_n + m$ is prime. Then, m must also be prime.

(b) (Euler brick) There exists an $a \times b \times c$ rectangular prism in which the distance between any two vertices is an integer.

(c) (No-three-in-line problem) Let n be an integer greater than 1. It is always possible to place $2n$ pawns on an $n \times n$ chessboard such that no three are collinear.

(d) (Brocard's problem) There exists an integer $n > 7$ for which $n! + 1$ is a square.

(e) (Erdős-Straus conjecture) For every $n \geq 2$, there are positive integers x, y, z such that

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

(f) (Inscribed square problem) On every closed curve in the plane, one can find four points that form a square.

Team Name: ____________________ Answer: ____________________

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11. **[11]** Let a, b, and c be real numbers such that

$$\frac{2a+b+c}{a} = 3, \quad \frac{a+2b+c}{b} = 4, \quad \frac{a+b+2c}{c} = N.$$

Compute N.

Team Name: ____________________ Answer: ____________________

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12. **[11]** Aditya and Noam start flipping coins at the same time. Aditya flips his coin every 15 seconds, but Noam is eating a banana, so he only flips his coin every 30 seconds. What is the probability that Noam flips heads before (not at the same time as) Aditya?

Team Name: ____________________ Answer: ____________________

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13. **[12]** Mr. Kats writes a 2024-digit sequence on the board consisting of zeroes, ones, and twos such that any four consecutive digits sum to a multiple of 3. How many sequences could he have written down?

Team Name: ____________________ Answer: ____________________

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14. **[12]** In the interior of a cube of side length 6, eight spheres are drawn, each of which is tangent to three of the cube's sides. Then, a ninth sphere is drawn tangent to the first eight spheres. If all nine spheres are congruent, what is their common radius?

Team Name: ____________________ Answer: ____________________

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15. **[13]** Compute the last 3 digits in the decimal representation of $2024^{2025^{2026}}$.

Team Name: ____________________ Answer: ____________________

. . . .

16. **[13]** In parallelogram $ABCD$, the reflection of diagonal $\overline{AC}$ over the bisector of $\angle BAD$ intersects $\overline{CD}$ at P. If $CD = 9$ and $DP = 4$, compute the length of $\overline{AD}$.

Team Name: ____________________ Answer: ____________________

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17. **[14]** Compute

$$\sum_{m=2}^{\infty}\sum_{n=3}^{\infty}\left(\frac{2}{n}\right)^m.$$

Team Name: ____________________ Answer: ____________________

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18. **[14]** Given a segment ℓ in the coordinate plane, we say that θ_ℓ is the smaller of the two angles formed by the line containing ℓ and the x-axis (measured in radians). Compute the sum of θ_ℓ over all segments ℓ whose endpoints have integer coordinates between 0 and 4, inclusive.

Team Name: ____________________ Answer: ____________________

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19. **[15]** For how many positive integers n less than 200 do n and $\binom{n}{3}$ have the same last two digits?

Team Name: ____________________ Answer: ____________________

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20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put Y if you think the statement is true, N if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.

(a) If the sum of the divisors of a positive integer n is prime, then n has a prime number of divisors.

(b) In Graphtopia, all flights between pairs of airports go in both directions. From each airport, there are flights to exactly three other airports, and one can travel between any two airports via a sequence of flights. Then, if a random airport is destroyed by a hurricane, it is guaranteed that one can still travel between any pair of intact airports.

(c) There exists an injective function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x^2 - 2023x) - f(2x - 2024)^2 \geq \frac{1}{4}$ for all real x.

(d) If a triangle has integer side lengths and integer area, then its area is even.

(e) Given triangle $\triangle ABC$, there is a unique parabola tangent to $\overline{AB}$ at B and $\overline{AC}$ at C.

(f) Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers for which $a_1 + a_2 + a_3 + \cdots$ converges. Suppose that, as n varies, the expression

$$\frac{a_n}{a_{n+1} + a_{n+2} + a_{n+3} + \cdots} = \frac{a_n}{\sum_{k=n+1}^{\infty} a_k}$$

is constant. Then, $a_1, a_2, a_3, \ldots$ is a geometric sequence.

Team Name: ____________________ Answer: ____________________

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21. **[16]** Given a prime p, we say that positive integer n is pseudo-cyclic with signature p if n, along with all the numbers formed by cyclically permuting the digits of n (preserving leading zeroes), are multiples of p. For example, 1034 is pseudo-cyclic with signature 11 because 1034, 0341, 3410, and 4103 are all multiples of 11.

    If n is a six-digit pseudo-cyclic number, compute the sum of all possible values of its signature.

Team Name: ____________________ Answer: ____________________

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22. **[16]** If x, y, z are positive real numbers such that $xyz(x+y+z) = 2024$, find the smallest possible value of $(x+y)(y+z)$.

Team Name: ____________________ Answer: ____________________

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23. **[17]** Let S_n be the set of points in the coordinate plane with nonnegative integer coordinates summing to at most n. For integers $n \geq 1$, let a_n be the number of ways to choose n points in S_n such that no two share an x- or a y-coordinate. What is $a_1 + a_2 + \cdots + a_{10}$?

Team Name: ____________________ Answer: ____________________

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24. **[17]** In convex quadrilateral $ABCD$, the lengths of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are 4, 6, and 2 respectively, and the measure of $\angle ABC$ is 60°. Let T be the point such that $BT = 4$ and $CT = 2$. When the circumradius of $\triangle ADT$ is minimal across all possible choices of D, what is the area of $ABCD$?

Team Name: ____________________ Answer: ____________________

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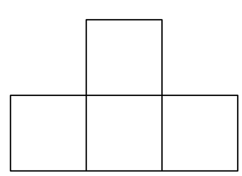
25. **[18]** Let N be the number of ways to tile a 4×2024 board using only T-tetrominoes (pictured below). Find the sum of the (not necessarily distinct) primes in the prime factorization of N. For example, if $N = 12$, submit $2 + 2 + 3 = 7$.



Team Name: ____________________ Answer: ____________________

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26. **[18]** Each face of a cube is labeled with a nonnegative integer. Then, each vertex of the cube is assigned the product of the numbers on the three faces containing that vertex. In how many possible ways can we choose the face labels so that the sum of the vertex labels is 81, where rotations and reflections of labelings are considered distinct?

Team Name: ____________________ Answer: ____________________

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27. **[19]** A regular 2024-gon $A_1A_2\cdots A_{2024}$ is inscribed in the unit circle in the complex plane. Given that A_1, A_2, $\ldots$, A_{1012} have nonnegative imaginary parts, the difference between the largest and smallest possible value of the sum of their imaginary parts can be expressed as $\tan\theta$, with $0 < \theta < \frac{\pi}{2}$. Compute θ.

Team Name: ____________________ Answer: ____________________

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28. **[19]** In acute triangle $\triangle ABC$, altitudes $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are drawn. Point X is on $\overline{BC}$ such that the line through X perpendicular to $\overline{BC}$ bisects $\overline{EF}$. Given that the distance from B to $\overline{DF}$ is 20, the distance from C to $\overline{DE}$ is 24, and the length of $\overline{BC}$ is 2024, compute $CX - BX$.

Team Name: ____________________ Answer: ____________________

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29. **[20]** How many pairs of nonnegative integers (n, k) are there such that $n \geq k$, $n + k < 64$, and $\binom{n}{k}$ is odd?

Team Name: ____________________ Answer: ____________________

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30. **[20]** Compute the number of pairs of natural numbers (a, b) such that $a, b < 1000$ and

$$\frac{a}{b+1} < \sqrt{3} < \frac{a+1}{b}.$$

Note that $\sqrt{3} \approx 1.732$.

Team Name: ____________________ Answer: ____________________

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