New York City Team Contest: Problems

Spring 2022

1. **[6]** Suppose $T = TNYWR$, where T, N, Y, W, and R are distinct integers. What is T?

   Team Name: ______________ Answer: ______________

2. **[6]** Find the smallest positive integer that is one more than a multiple of 6, and is neither a prime nor a perfect square.

   Team Name: ______________ Answer: ______________

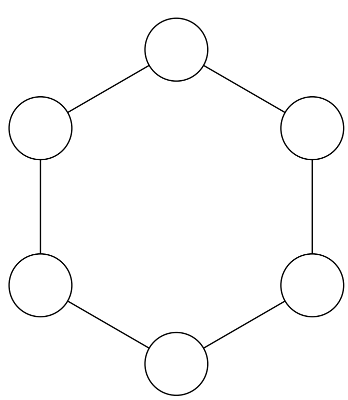
3. **[7]** What is the minimum number of coin flips to guarantee you can find a consecutive string of HH, TT, or $HTHT$ (where H represents flipping a heads, and T represents flipping a tails)?

   Team Name: ______________ Answer: ______________

4. **[7]** Equilateral triangle ABC has side length 22. A point P inside the triangle is chosen such that $PB = PC = 14$. Compute PA.

   Team Name: ______________ Answer: ______________

5. **[8]** Six circles are arranged into a hexagon, as shown, and the six numbers $1, 2, 3, 4, 5$, and 6 are put into these circles in some order. On each edge of the hexagon, Stanley writes down the sum of the two numbers in the circles connected to this edge, and Brian writes down the product of the two numbers in the circles connected to this edge. Stanley then notices that he wrote down only prime numbers. What is the sum of the numbers Brian wrote down?



   Team Name: ______________ Answer: ______________

6. **[8]** Mr. Cocoros collects dice. A die of "size n" has n faces labeled $1, 2, \ldots, n$, each equally likely to show up. Mr. Cocoros has all dice of size 4 to N for some positive integer N. Mr. Cocoros rolls all of his dice once. If the probability that Mr. Cocoros rolls at least one "1" is 75%, what is N?

Team Name: ____________ Answer: ____________

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7. **[9]** The first two numbers in a list of integers are 20 and 22, respectively. If the sum of the numbers in the list is equal to the product of the numbers in the list, what's the least possible number of entries in the list?

Team Name: ____________ Answer: ____________

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8. **[9]** Suppose three of the vertices of a cube are $(2, 4, 8), (12, 5, 3)$, and $(11, 7, 14)$. Compute the volume of this cube.

Team Name: ____________ Answer: ____________

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9. **[10]** Suppose $N = \underline{a}\,\underline{b}\,\underline{b}\,\underline{c}\,\underline{b}\,\underline{a}_3 = \underline{a}\,\underline{b}\,\underline{b}\,\underline{a}\,\underline{c}_4$, (these are the base three and base four representations of N) where a, b, and c represent different digits. Compute N in base 5.

Team Name: ____________ Answer: ____________

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10. **[Up to 10] Pick a number between** 0 **and** 1**, inclusive.** Let σ denote the standard deviation of all valid submissions. Estimate 2σ. (Not sponsored by the way.)

If your submission is X and the actual value of 2σ is Y, you will receive

$$\max\left(\left\lfloor \frac{60}{|X-Y|+1} - 50 \right\rfloor, 0\right)$$

points.

Team Name: ____________ Answer: ____________

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11. **[11]** Compute the positive integer n satisfying

$$(n-16)^2 + (n-15)^2 + \cdots + (n-1)^2 + n^2 = (n+1)^2 + (n+2)^2 + \cdots + (n+15)^2 + (n+16)^2.$$

Team Name: ____________ Answer: ____________

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12. **[11]** Let $\triangle ABC$ be an isosceles triangle with vertex A and $\angle BAC = 80°$. Let D be on segment BC such that $\angle BAD = 65°$, and let E be on segment AC such that $\angle EBC = 30°$. Compute $\angle BED$.

Team Name: ____________ Answer: ____________

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13. **[12]** How many ordered triples of positive integers (a, b, c) are there such that $(a^a \times b^b \times c^c) \mid 6^6$?

(We say "$x \mid y$" if x is a divisor of y.)

Team Name: ______________ Answer: ______________

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14. **[12]** I was born on May 10, 2004. My dad was born on July 13, 1968. Presuming both of us live until 100, for how many days will my dad be twice my age?

Team Name: ______________ Answer: ______________

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15. **[13]** Let $\triangle ABC$ be a triangle such that $AB = 3$, $BC = 4$, and $CA = 5$. Let D be the point on BC such that $AD = DC$, and let the circumcircle of $\triangle ADC$ intersect AB at $E \neq A$. Compute EC.

Team Name: ______________ Answer: ______________

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16. **[13]** Compute

$$(\log_2(20))(\log_2(40))(\log_2(640)) - (\log_2(10))(\log_2(160))(\log_2(320)).$$

Team Name: ______________ Answer: ______________

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17. **[14]** Adi, Das, and Big J are playing in a 3-person tournament of ping pong. Each of them is "equally good" at ping pong, so in any given game, the two players of that game are equally likely to win. After each game, the winner stays on the table, while the loser rotates off. The winner of the tournament is the first person to win 2 games in a row. Given that Adi and Das play the first match, what is the probability that Big J defies the odds to win the tournament?

Team Name: ______________ Answer: ______________

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18. **[14]** A circle Ω has center O and radius 8. Let A be a point on Ω, and B be the midpoint of OA. A line through B is drawn intersecting Ω at X and Y such that $\frac{BX}{BY} = 2$. Compute the area of $\triangle AXY$.

Team Name: ______________ Answer: ______________

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19. **[15]** Suppose n is a randomly and uniformly selected 56-digit positive integer. Compute the probability that the sum of the digits of $n + 56$ is greater than the sum of the digits of n.

Team Name: ______________ Answer: ______________

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20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ *points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.*

(1) Four positive integers form a nonconstant geometric sequence. Then their sum must have at least four positive integer factors.

(2) It is known that $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$. There is some $k > 3$ for which $1^k + 2^k + \cdots + n^k$ is equal to $f(n)^2$ for some polynomial f with rational coefficients.

(3) A triangle has integer side lengths and integer area. Its side lengths form an arithmetic sequence with common difference 4. Then its perimeter must be be a multiple of 8.

(4) All real polynomials P satisfying $P(x^2) = P(x)^2$ for all real x are either 0 or x^n for some nonnegative integer n.

(5) Let $\sigma(n)$ denote the sum of the divisors of n. Then

$$\sum_{k=1}^{n} \sigma(k) \leq n^2$$

for all positive integers n.

(6) Consider a triangle divided into 6 smaller triangles by 3 concurrent lines passing through its vertices. There exists such a configuration where the 6 triangles created have areas that form a non-constant geometric sequence in some order.

Team Name: ____________ Answer: ____________

21. **[16]** Define $F_0 = 0, F_1 = 1$, and for $n \geq 1$, $F_{n+1} = F_n + F_{n-1}$. Compute the sum of the prime factors of

$$\sum_{n=1}^{10} F_{n+3}F_n.$$

Team Name: ____________ Answer: ____________

22. **[16]** In convex quadrilateral $ABCD$, $\angle A = 37°$, $\angle B = 23°$, $AB = 22$, $CD = 20$, and $AD = BC$. Compute $[ABCD]$, the area of quadrilateral $ABCD$.

Team Name: ____________ Answer: ____________

23. **[17]** How many ordered pairs of positive integers (a, b) are there such that $a, b \leq 50$ and

$$\tau(ab) = \tau(a) + \tau(b) - 1,$$

where $\tau(n)$ is the number of divisors of n?

Team Name: ____________ Answer: ____________

24. **[17]** Let $f(n)$ denote the number of ordered pairs of positive integers (a, b) satisfying $ab = n$ and $a, b \leq 20$. Compute

$$\sum_{n=1}^{\infty} (-1)^n f(n) n.$$

Team Name: ____________ Answer: ____________

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25. **[18]** There is a unique quadratic function f such that

- there exists a constant c such that $f(f(x)) = c$ is satisfied only by $x \in \{1, 2, 3\}$, and
- $f(0) = 23$.

Compute c.

Team Name: ____________ Answer: ____________

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26. **[18]** Let the incircle of triangle ABC touch sides AC and AB at E and F, respectively. If $EF = 42$ and line EF bisects arcs $\widehat{AC}$ and $\widehat{AB}$ of the circumcircle of triangle ABC, find the area of quadrilateral $AEIF$.

Team Name: ____________ Answer: ____________

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27. **[19]** Suppose that $P(x)$ is a polynomial of degree at most 5 such that $P(2^k) = k$ for $k = 1, 2, 3, 4, 5$, and 6. Compute $P(2^7)$.

Team Name: ____________ Answer: ____________

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28. **[19]** In a square grid with 11 rows and 10 columns, every square not in the top row is randomly assigned a right arrow or an up arrow. You can only move from one square to the adjacent square where the arrow is pointing. What is the probability that there is a path from some square in the bottom row to some square in the top row?

Team Name: ____________ Answer: ____________

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29. **[20]** Define

$$f(n) = \sum_{wxyz \mid n} \frac{1}{wxyz},$$

where the summation runs over all ordered quadruples of positive integers (w, x, y, z) satisfying $wxyz \mid n$. Let the set A consist of numbers of the form $2^p 3^q 7^r$, where p, q, and r are nonnegative integers.

Find the smallest value of M satisfying $f(x) \leq M$ for all $x \in A$.

Team Name: ____________ Answer: ____________

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30. **[20]** Let ω and Γ be circles with radii 15 and 25, respectively, such that the centers of ω and Γ are 30 apart. Say that ω and Γ have common external tangent AB with A on ω, B on Γ, and say that ω and Γ intersect at C and D with C closer to AB than D. Let F be the reflection of C across AB. Compute DF.

Team Name: ______________ Answer: ______________

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