New York City Team Contest: Problems

Fall 2021

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1. **[6]** Compute

$$\left(\frac{1}{1+2+4+8+16+32}\right) \cdot \left(\frac{1}{1}+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}\right).$$

Team Name: _____________ Answer: _____________

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2. **[6]**

We're no strangers to love
You know the rules and so do I
A full commitment's what I'm thinking of
You wouldn't get this from any other guy

I just wanna tell you how I'm feeling
Gotta make you understand

Never gonna give you up
Never gonna let you down
Never gonna run around and desert you
Never gonna make you cry
Never gonna say goodbye
Never gonna tell a lie and hurt you

Team Name: _____________ Answer: _____________

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3. **[7] The answer to this question is equal to the product of the answers to questions 1, 2, and 3.**

Suppose problem k on NYCTC is worth $\lfloor \frac{11+k}{2} \rfloor$ points, except for problem 20, which is worth 28 points. Find the sum of all n such that problem n is worth n points.

(Here, $\lfloor x \rfloor$ is equal to the largest integer that is at most x. For example, $\lfloor \pi \rfloor = 3$ and $\lfloor 13 \rfloor = 13$.)

Team Name: _____________ Answer: _____________

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4. [**7**] The number $\underline{A}\,\underline{0}\,\underline{2}\,\underline{0}\,\underline{2}\,\underline{1}$ is divisible by 7, where A is a digit. What is A?

Team Name: ______________ Answer: ______________

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5. [**8**] Square $ABCD$ and line ℓ lie on a plane. Let X and Y be the feet of the perpendiculars from A and C to line ℓ, respectively. If $AX = 8, CY = 6$, and $XY = 4$, compute the area of $ABCD$.

Team Name: ______________ Answer: ______________

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6. [**8**] A fair coin is flipped 7 times. What is the probability the product of the number of heads and the number of tails is a multiple of 3?

Team Name: ______________ Answer: ______________

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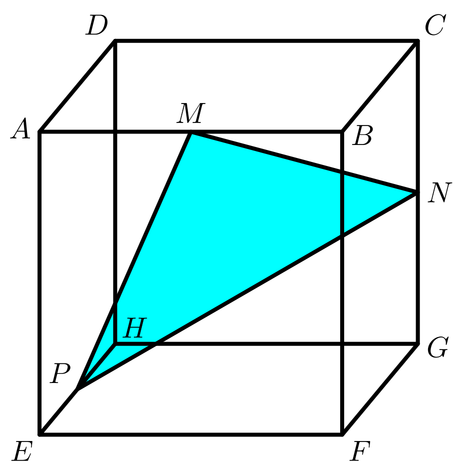
7. [**9**] Suppose $ABCDEFGH$ is a cube of side length 4, as shown. Suppose M, N, and P are the midpoints of AB, CG, and HE, respectively. What is the area of $\triangle MNP$?



Team Name: ______________ Answer: ______________

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8. [**9**] There are 10 lily pads arranged in a circle. Kelvin the frog is currently on a lily pad, and every minute either jumps 2 lily pads clockwise or 3 lily pads counterclockwise with equal probability. After 5 minutes, what is the probability Kelvin is back where he started?

Team Name: ______________ Answer: ______________

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9. [**10**] Suppose $225 \cdot 226 + 1 = pq$ for primes $p > q$. Compute $2p + q$.

Team Name: ______________ Answer: ______________

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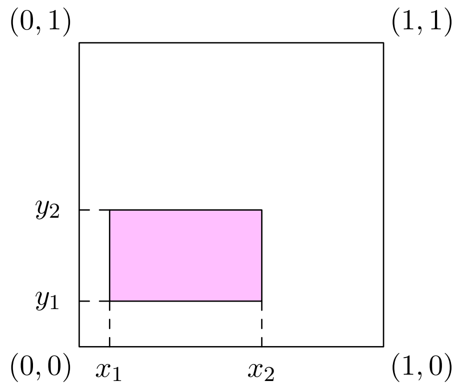
10. **[Up to 10]** Pick a rectangle fully contained within the unit square $\mathcal{S} = [0,1] \times [0,1]$ and with sides parallel to $\mathcal{S}$.

Let T be the number of submissions. Let I be the number of rectangles other teams submit that intersect with yours (sharing an edge or vertex counts as intersection). Let A be the area of your rectangle. Your score is $\lfloor cA(T-I-1) \rfloor$, where $c = \frac{10}{\max(T-I-1)A}$ over all submissions.

Your answer should be submitted in the form (x_1, x_2, y_1, y_2), where your rectangle is $[x_1, x_2] \times [y_1, y_2]$ (in particular, your rectangle will have vertices $(x_1, y_1), (x_1, y_2), (x_2, y_1)$, and (x_2, y_2)). Your submission should satisfy $0 \leq x_1 < x_2 \leq 1$ and $0 \leq y_1 < y_2 \leq 1$. Invalid submissions will result in 0 points.



Team Name: ______________ Answer: ______________

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11. **[11]** A group of 103 people, including Taylor and Kanye, will form a 5-person team. The captain of the team (who is a member of the team) is either Taylor or Kanye. If Taylor is the captain, Kanye refuses to also be on the team. However, if Kanye is the captain of the team, then Taylor is okay with being on the team. If a team is randomly selected from all possible teams, compute the probability Kanye is on the team.

Team Name: ______________ Answer: ______________

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12. **[11]** There is a unique triple of primes (p, q, r) satisfying $2pqr - 5p^2 - 5q^2 + 5r^2 = 0$ and $p < q$. Find (p, q, r).

Team Name: ______________ Answer: ______________

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13. **[12]** Suppose M and N are the midpoints of sides AB and AC of $\triangle ABC$. Let line MN intersect the circumcircle of $\triangle ABC$ at X and Y such that M is between X and N. If $XM = 5, MN = 11$, and $NY = 9$, compute the area of $\triangle ABC$.

Team Name: ______________ Answer: ______________

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14. **[12]** How many ways can the cells of a 3×3 grid be colored blue and red such that no row or column has all three of its cells of the same color?

Team Name: ______________ Answer: ______________

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15. **[13]** Let ℓ be the A-angle bisector of triangle ABC. Let the feet from B and C to ℓ be D and E, respectively. If $BD = 12, CE = 24$, and $DE = 15$, find the area of triangle ABC.

Team Name: ____________ Answer: ____________

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16. **[13]** Suppose a and b are positive integers satisfying $a + b = 210$ and

$$6 \cdot \gcd^2(a, b) + \operatorname{lcm}^2(a, b) = 7ab.$$

Find the sum of all possible values of $|a - b|$.

Team Name: ____________ Answer: ____________

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17. **[14]** Compute $\displaystyle\sum_{k=1}^{49} \sqrt{k - \sqrt{k^2 - 1}}$.

Team Name: ____________ Answer: ____________

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18. **[14]** Triangle ABC has perimeter 5. If $\angle BAC = 60^\circ$ and $AB^3 + AC^3 = 12$, compute $AB \cdot AC$.

Team Name: ____________ Answer: ____________

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19. **[15]** If a, b, and c are positive reals that are not all the same satisfying

$$a^2 + b^2 + c = b^2 + c^2 + a = c^2 + a^2 + b,$$

then find the maximum possible value of abc.

Team Name: ____________ Answer: ____________

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20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.

(1) There exists a perfect square which is equal to the sum of the squares of five consecutive positive integers.

(2) A square $ABCD$ is completely covered by finitely many (possibly overlapping) disks. The sum of the radii of these disks must be at least $\frac{AC}{2}$.

(3) Suppose a, b, c, d, and e are integers satisfying

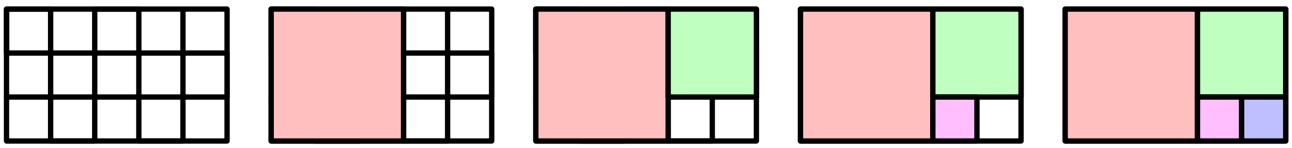
$$a + b = c + d + e,$$
$$a^2 + b^2 = c^2 + d^2 + e^2, \text{ and}$$
$$a^3 + b^3 = c^3 + d^3 + e^3.$$

Then $abcde = 0$.

(4) There exists an infinite set of positive integers S such that for any positive integers a and b, neither of which divides the other, at least one, but not all, of the integers $\gcd(a,b), a, b, \operatorname{lcm}(a,b)$ are elements of S.

(5) A rectangle has integer side lengths. We tile the rectangle with squares greedily; this means we first draw the largest possible square inside this rectangle (and if multiple exist, pick the top-left most one), which results in a smaller rectangle. We then repeat the process, tiling the rectangle with squares.

An example of a greedy tiling of a 3×5 rectangle is shown below.



This greedy method is the optimal way to tile any starting rectangle with squares (i.e. it uses the least number of squares out of any tiling of the rectangle with squares).

(6) Suppose O, H, and I are the circumcenter, orthocenter, and incenter of the acute, scalene $\triangle ABC$. The circumcircle of $\triangle OIH$ must pass through an even number of vertices of $\triangle ABC$.

Team Name: ____________ Answer: ____________

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21. **[16]** Akash and Kimi are playing a game. Akash thinks of a polynomial $P(x)$ of degree 2021 with nonnegative integer coefficients, all at most 2021. A *move* is when Kimi gives Akash a real number r, and Akash tells Kimi $P(r)$. At most how many moves must Kimi make to determine the polynomial $P(x)$?

Team Name: ____________ Answer: ____________

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22. **[16]** Let a, b, c, d be positive real numbers such that

$$\begin{aligned} a^2 + b^2 &= 1, \\ b^2 + c^2 &= 16, \\ c^2 + d^2 &= 64, \text{ and} \\ ac &= bd. \end{aligned}$$

Compute $ab + bc + cd + da$.

Team Name: ____________ Answer: ____________

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23. **[17]** A positive integer n is *similvisible* if it has only single-digit prime factors and $4n, 5n, 6n$, and $7n$ have the same number of positive integer divisors. Compute the sum of the reciprocals of all similvisible integers.

Team Name: ____________ Answer: ____________

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24. **[17]** Suppose $(\sigma_1, \sigma_2, \ldots, \sigma_{2021})$ is a permutation of $(1, 2, 3, \ldots, 2021)$. Suppose, across all such permutations, m is the minimum value of the expression

$$|\sigma_2 - \sigma_1| + |\sigma_3 - \sigma_2| + \cdots + |\sigma_{2021} - \sigma_{2020}| + |\sigma_1 - \sigma_{2021}|$$

and suppose n is the number of such permutations such that the given expression is equal to m. Compute n.

Team Name: ____________ Answer: ____________

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25. **[18]** Let triangle ABC be such that $AB = 7$ and $AC = 8$. There exists a point D on segment $\overline{BC}$ such that $AD = 6$ and the inradius of triangle ABD is equal to the inradius of triangle ACD. Find BC.

Team Name: ____________ Answer: ____________

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26. **[18]** Let $m = 2^4 \cdot 3^4$. Suppose k is a randomly selected integer from 1 to m, inclusive. Let ℓ be the expected value of $\log_{10}(\gcd(k, m))$. Find the number of not necessarily distinct prime factors of $10^{\ell m}$. (For example, $12 = 2^2 \cdot 3$ has 3 not necessarily distinct prime factors.)

Team Name: ____________ Answer: ____________

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27. **[19]** Suppose c is a real number such that when the roots of $x^3 + 3x^2 + 12x + c$ are plotted in the complex plane, they form a non-degenerate triangle with orthocenter at the origin. Compute c.

Team Name: ____________ Answer: ____________

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28. **[19]** Alex writes the ordered pair $(1, 0)$ on a chalkboard. Every minute, he randomly and uniformly chooses two integers c and d, between 1 and 5, inclusive. Then, if at that time he has the ordered pair (a, b) on the board, he erases it and writes the ordered pair $(ac + bd, ad + bc)$ on the board. Find the expected number of minutes it will take for both of the numbers in his ordered pair to be divisible by 5.

Team Name: ____________ Answer: ____________

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29. **[20]** Stan the Cat is a 2021-dimensional creature, playing with 2021-dimensional hyperspherical marbles $\Omega_1, \Omega_2, \ldots, \Omega_{2023}$. The radii of $\Omega_1, \Omega_2, \ldots, \Omega_{2021}$ are 1, while the radii of Ω_{2022} and Ω_{2023} are r. Stan then arranges his marbles such that they are all pairwise externally tangent. Find r.

Team Name: ____________ Answer: ____________

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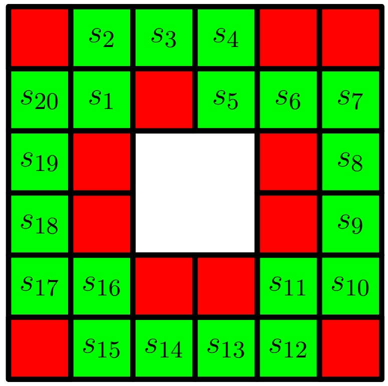
30. **[20]** A 32×32 grid has the middle 28×28 grid cut out, forming a "donut" of width 2. From this donut, a subset of cells of size m is called *m-cool* if they can be labelled $s_1, s_2, \ldots, s_m$ such that s_i and s_j share exactly one side if and only if $i - j \equiv \pm 1 \pmod{m}$. Let M be the maximum positive integer such than an M-cool set exists, and suppose there are N distinct M-cool sets. Compute N.

The green cells below are an example of a 20-cool set on a smaller donut (a 2×2 removed from a 6×6).



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