NYCMT 2024-2025 Homework #1

NYCMT

September 20 - September 27, 2024

These problems are due September 27.

Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Write down any progress you make on problems you don't solve. Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on September 27, you can scan your solutions and email them to ashleyzhu111@gmail.com, sjschool26@gmail.com, and stevenytlou@gmail.com. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or using one of the above emails.

Problems are NOT difficulty-ordered, so you should read and try all of them.

Enjoy!

Problem 1. Consider complex polynomials $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ with the zeroes $x_1, x_2, \cdots, x_n$ and $Q(x) = x^n + b_1x^{n-1} + \cdots + b_n$ with the zeroes $x_1^2, x_2^2, \cdots, x_n^2$. Prove that if $a_1 + a_3 + a_5 + \cdots$ and $a_2 + a_4 + a_6 + \cdots$ are real numbers, then $b_1 + b_2 + b_3 + \cdots + b_n$ is also real.

Problem 2. Ashley and Sophia are playing a game. They take turns flipping a coin, with Ashley going first, and keep track of the total number of heads they have flipped. Whoever reaches 2 heads flipped first wins. What is the probability that Sophia wins? (Note: The two heads need not be consecutive.)

Problem 3. Let N be the number of ordered triples of positive integers (a, b, c) are such that $\operatorname{lcm}(a, b, c) = 20!$ and $\gcd(a, b, c) = 1$. Find the number of divisors of N.

Problem 4. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$ and $\angle BAC = 100°$. Let point D be on side AC such that BD bisects $\angle ABC$. Prove that $AD + DB = BC$.

Problem 5. Let $n \in \mathbb{N}$ and $a_1 < a_2 < a_3 < \cdots < a_{\phi(n)}$ be the integers less than n and relatively prime to n. Prove that $a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)} \equiv \pm 1 \pmod{n}$ for $n \geq 2$.
Bonus. For which n is $a_1 \cdot a_2 \cdot a_3 \cdots a_{\phi(n)} \equiv -1 \pmod{n}$?