NYCMT SHRIMP Solutions

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Problem 1. Andrew and Jaemin are expert painters, and each of them can create one painting in five days. However, when working together, they distract each other, with Andrew working at half of his normal pace, and Jaemin working at a third of his normal pace. How many days would it take the two of them, working together, to create one painting? (Assume Andrew and Jaemin work at constant rates).

Answer. $\boxed{6}$

Solution. When working at their normal paces, Andrew and Jaemin can each create $\frac{1}{5}$ of a painting each day. When distracted, Andrew creates $\frac{1}{2} \cdot \frac{1}{5} = \frac{1}{10}$ and Jaemin creates $\frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}$ of a painting each day instead. Working together, they can create $\frac{1}{10} + \frac{1}{15} = \frac{1}{6}$ of a painting each day, which means it would take them $\boxed{6}$ days to create one painting. □

Problem 2. Taylor Swift has 4 spaces in a row on which she can place her 10 indistinguishable cookies. If every cookie must be placed on one of the 4 spaces, and any two spaces with cookies must have at least one blank space *baby* between them (meaning two spaces with cookies cannot be adjacent), in how many possible ways can she place her cookies?

Answer. $\boxed{31}$

Solution. We do casework on how many spaces contain cookies.

Case 1: Exactly one space contains cookies. Then, it must contain all ten, and there are four spaces, so there are 4 placements for this case.

Case 2: Exactly two spaces contain cookies. Then, they cannot be adjacent, so there are three choices for the two spaces. Then, each space must contain at least one cookie, and as a result, there are eight cookies left to split among two spaces. This gives $\binom{9}{1} = 9$ ways, and $3 \cdot 9 = 27$ placements for this case.

We note that it is impossible to place cookies on more than two spaces, because at least two would be adjacent.

The answer is $4 + 27 = \boxed{31}$. □

Problem 3. The numbers $1, 2, 3, \ldots, 7$ are partitioned into two groups. Let the product of the numbers in the first group be A and the product of the numbers in the second group be B. Find the minimum possible value of $|A - B|$.

Answer. $\boxed{2}$

Solution. We know that the product $AB = 7! = 5040$ is fixed. In order to minimize the absolute difference between A and B, we want each of them to be as close as possible to $\sqrt{5040} \approx \sqrt{5041} = 71$. We now note that $5040 = 71^2 - 1^2 = 70 \cdot 72$, and it is possible to achieve this if the first group contains 2, 5, and 7, and if the second group contains 3, 4, and 6. It is clear that no smaller value of $|A - B|$ is possible, so our answer is $|70 - 72| = \boxed{2}$. □

Problem 4. Square $ABCD$ has area 16. Point P is chosen in the plane such that $[PAB] = 4$, $[PCD] = 12$, and $PA = PC = x$. Find x.

Answer. $\boxed{2\sqrt{10}}$

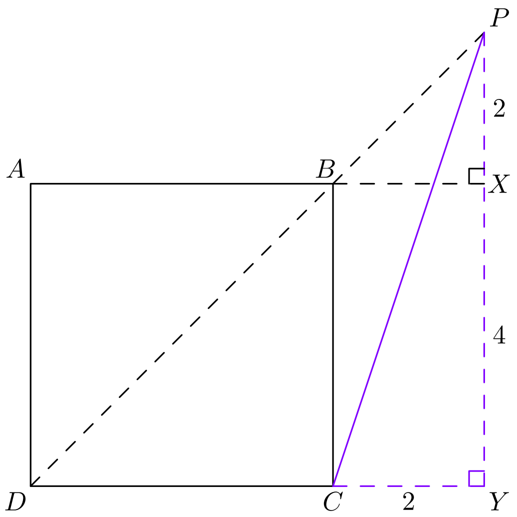
Solution. Square $ABCD$ has side length $\sqrt{16} = 4$. In order to compute the areas of $\triangle PAB$ and $\triangle PCD$, we drop the altitudes from P to $\overline{AB}$ and $\overline{CD}$, intersecting those segments at X and Y respectively. We now have

$$\begin{aligned}[PAB] &= \frac{1}{2} \cdot PX \cdot AB \\ 4 &= \frac{1}{2} \cdot PX \cdot 4 \\ PX &= 2\end{aligned}$$

and

$$\begin{aligned}[PCD] &= \frac{1}{2} \cdot PY \cdot CD \\ 12 &= \frac{1}{2} \cdot PY \cdot 4 \\ PY &= 6\end{aligned}$$

from the given triangle areas. Now, there are two cases to consider. If P is inside $ABCD$, then the altitudes to opposite sides must add up to the side length, 4. If P is outside $ABCD$, then the absolute difference between the altitudes must equal the side length, 4. It is clear that $6 - 2 = 4 \neq 6 + 2$, so P is outside $ABCD$ and closer to $\overline{AB}$ then $\overline{CD}$, as shown in the diagram below. We also know that $PA = PC$, meaning P is on the perpendicular bisector of $\overline{AC}$, or, equivalently, line $\overleftrightarrow{BD}$, making $PX = BX = CY = 2$.



This now uniquely determines P, and x can be computed with the Pythagorean Theorem in $\triangle PYC$. Our answer is $x = \sqrt{PY^2 + YC^2} = \sqrt{6^2 + 2^2} = \sqrt{40} = \boxed{2\sqrt{10}}$. $\square$

Problem 5. The ordered pair of positive integers (x, y) satisfies

- $\gcd(x, y) = 11$
- $\gcd(2023x, y) = 187$

Find the smallest possible value of $x + y$.

Answer. $\boxed{198}$

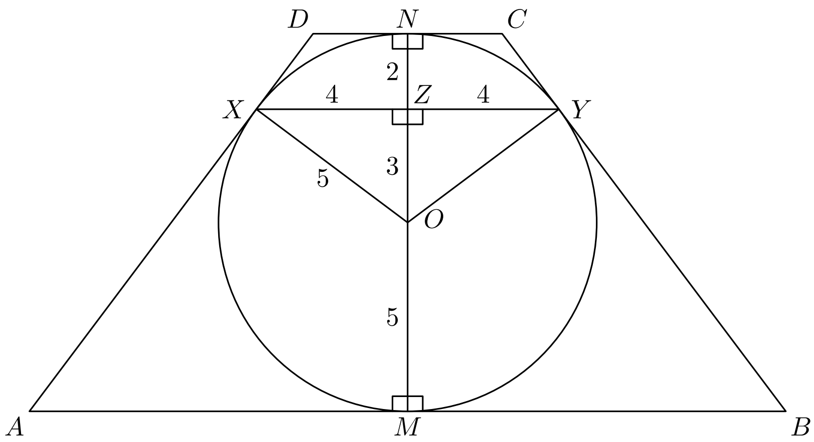
Solution. The first condition implies $11 \mid x, y$ and the second implies $187 \mid 2023x, y$. However, since $17 \mid 2023$ and $11 \mid 187$, we only need $11 \mid x$ and $187 \mid y$, and the minimal pair $(x, y) = (11, 187)$ satisfies the conditions, so our answer is $11 + 187 = \boxed{198}$.

Remark. $11 \mid x$ and $187 \mid y$ are necessary, but not sufficiently equivalent to the given conditions. For example, it is impossible for any prime $p \neq 11$ or any power 11^n with $n > 1$ to divide both x and y, or their greatest common factor would not be 11, but a larger multiple of 11. Finding the minimal pair, however, implicitly satisfies these constraints as well, and as a result, they do not need to be explicitly considered. □

Problem 6. A circle of radius 5 is inscribed in an isosceles trapezoid. The distance between the tangency points of the circle with the legs of the trapezoid is 8. Find the area of the trapezoid.

Answer. $\boxed{125}$

Solution. Because the circle is tangent to both parallel bases of the trapezoid, the height of the trapezoid is equal to the circle's diameter, 10.

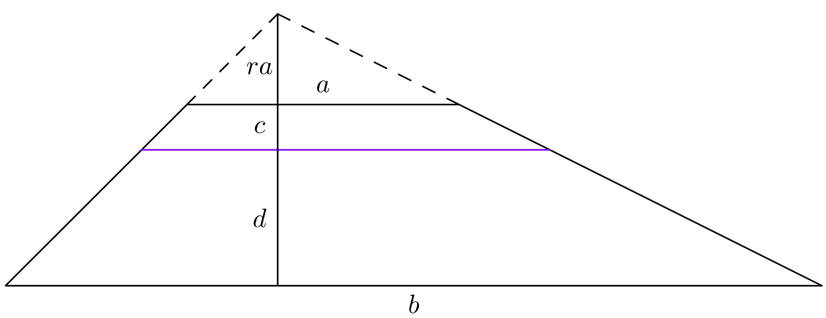


We are given $XY = 8$. Z is the midpoint of $\overline{XY}$ by symmetry, so $XZ = YZ = \frac{1}{2}XZ = 4$ and $OZ = \sqrt{OX^2 - XZ^2} = \sqrt{5^2 - 4^2} = 3$. As a result, $MZ = OM + OZ = 5 + 3 = 8$ and $NZ = MN - MZ - 10 - 8 = 2$.

Now, in order to find the area, we want to find the sum of the bases. Let $AB = 2x$ and $CD = 2y$. We can relate the bases using their relative distances from $\overline{XY}$.

Claim. If a segment parallel to the bases of a trapezoid splits the trapezoid so that it is c away from the base of length a and d away from the base of length b, its length is $\frac{ad+bc}{c+d}$.

Proof. Extending the legs of the trapezoid to their intersection forms three triangles, which are pairwise similar by parallel lines. This means that the ratio of the lengths of their heights to the lengths of their bases is constant; let this ratio be r.



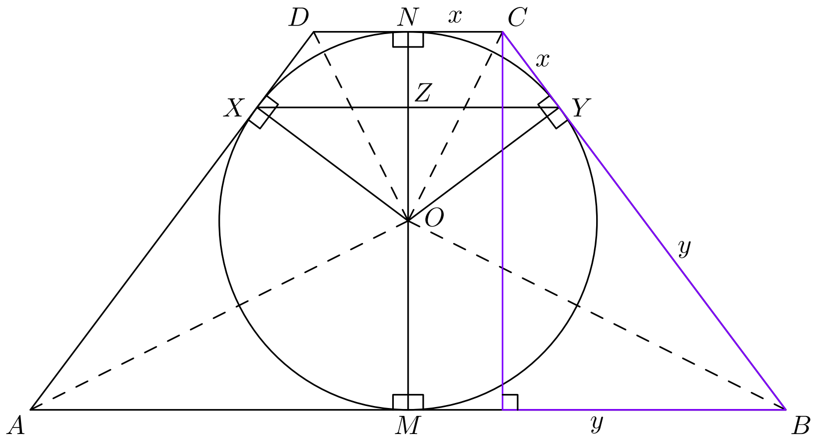
Then, the shortest height is ra, the longest height is $ra + c + d = rb \implies r = \frac{c+d}{b-a}$, and the length in question is $\frac{ra+c}{r} = a + \frac{c(b-a)}{c+d} = \frac{ad+bc}{c+d}$ as desired.

Applying this result to our trapezoid gives

$$8 = XY = \frac{CD \cdot MZ + AB \cdot NZ}{MN} = \frac{8 \cdot 2y + 2 \cdot 2x}{10}$$

and $x + 4y = 20$. We now have one equation.

Because not all isosceles trapezoids have incircles, the existence of one implicitly gives us information.



Specifically, note that $\triangle OYC \cong \triangle ONC$ and $\triangle OYB \cong \triangle OMB$ by HL congruence (and symmetrically on the other side), as all radii of the incircle are congruent. This means $BM = BY = x$ and $CN = CY = y$, and we can apply the Pythagorean Theorem: $(y - x)^2 + 10^2 = (x + y)^2$, giving us the second equation $xy = 25$.

Solving the system produces $x = \frac{5}{2}$ and $y = 10$, meaning $CD = 5$ and $AB = 20$. Finally, the area is $\frac{1}{2} \cdot 10 \cdot (5 + 20) = \boxed{125}$. $\square$

Problem 7. p and q are primes such that $p+q$ is prime and p^2+q is prime. Compute all possible values of p^3+q.

Answer. $\boxed{\{11, 29\}}$

Solution. Since $p+q$ must be an odd prime (as 2 is too small to be achieved), p and q cannot both be odd or both be even. The only even prime is 2, so one of p and q is 2. This gives us two cases.

Case 1: $p = 2$. Then, both $q+2$ and $q+4$ are primes greater than 3, and therefore cannot be divisible by 3. The first statement implies that q cannot be 1 more than a multiple of 3, and the second implies that q cannot be 2 more than a multiple of 3. As a result, $3 \mid q$ and q must equal 3, as 3 is the only prime divisible by 3.

This case produces $p^3 + q = 2^3 + 3 = 11$.

Case 2: $q = 2$. Then, $p^2 + 2$ is a prime greater than 3. If $3 \nmid p$, then p^2 is 1 more than a multiple of 3, and $p^2 + 2$ is divisible by 3, and therefore cannot be prime. So, $3 \mid p$, and $p = 3$ following similar logic in Case 1.

This case produces $p^3 + q = 3^3 + 2 = 29$.

Our answer is the set $\boxed{\{11, 29\}}$.

Remark. This problem was originally going to be proposed as a homework problem, and the statement would have been modified from "Compute all possible values of p^3+q" to "Prove that p^3+q is prime" for extra effect.

□

Problem 8. Suppose a monic polynomial $f(x)$ of degree 3 satisfies the equation $f(2^n - 1) = 4^n - 1$ for $n = 1, 2, 3$. Find $f(0)$.

Answer. $\boxed{-21}$

Solution. Let $x = 2^n - 1$. Then, since $4^n - 1 = (2^n - 1)(2^n + 1)$, we have $f(x) = x(x+2)$ for $x = 1, 3, 7$. This means that $f(x) - x(x + 2)$ has 1, 3, and 7 as zeroes, and must be $(x - 1)(x - 3)(x - 7)$, as it is still a monic polynomial of degree 3. So, $f(x) = (x - 1)(x - 3)(x - 7) + x(x + 2)$ and $f(0) = \boxed{-21}$. □

Problem 9. The numbers $1, 2, \ldots, 8, 9$ are placed randomly into a 3×3 grid such that each number appears exactly once. What is the probability that each 2×2 square contains at least one perfect square?

Answer. $\boxed{\frac{4}{7}}$

Solution. Name the cells of the grid M for middle, S for side, and C for corner, as shown.

C	S	C
S	M	S
C	S	C

We will do casework on what types of cells contain perfect squares, and compute the total number of ways to satisfy the condition. The total number of placements, regardless of whether the condition is satisfied, is clearly $9!$.

Case 1: The middle cell contains a perfect square. Since every 2×2 subgrid contains the middle cell, the condition is automatically satisfied. There are three perfect squares to choose from, so there are $3 \cdot 8!$ placements in this case.

Case 2: The three perfect squares are all in side cells. It can be shown that this also automatically satisfies the condition, as it must be true that two side cells that are not touching both contain perfect squares, which covers all 2×2 subgrids. There are 4 ways to choose which side cell does not contain a perfect square, $3!$ ways to arrange the perfect squares, and $6!$ ways to arrange the rest, resulting in $24 \cdot 6!$ placements in this case.

Case 3: Two perfect squares are in side cells, and one is in a corner cell. Then, to satisfy the condition, the two side cells either do not touch, or they do, and the corner cell is forced to cover the missing subgrid. In the former case, there are 2 ways to choose the side cells, 4 to choose the corner cell, and $3! \cdot 6!$ to arrange the numbers, and in the latter, there are 4 ways to choose the side cells, the corner cell is forced, and also $3! \cdot 6!$ ways to arrange the numbers, resulting in $72 \cdot 6!$ placements in this case.

Case 4: Two perfect squares are in corners cells, and one is in a side cell. Then, depending on the chosen side cell, the corner cells are forced. This results in $4 \cdot 3! \cdot 6! = 24 \cdot 6!$ placements in this case.

Case 5: The three perfect squares are all in corner cells. Then, the condition is never satisfied, resulting in 0 placements in this case.

All cases have been covered, and we have a total of $3 \cdot 8! + 24 \cdot 6! + 72 \cdot 6! + 24 \cdot 6! = 288 \cdot 6!$ valid placements, and our desired probability is $\frac{288 \cdot 6!}{9!} = \frac{288}{9 \cdot 8 \cdot 7} = \boxed{\frac{4}{7}}$.

$\square$

Problem 10. A sequence a_n of integers is defined by $a_1 = 2$, $a_2 = 4$ and, for all integers $n \geq 3$,

$$\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{n-1}} + \frac{n}{a_n} = 1.$$

Find $a_{2023} - a_{2022}$.

Answer. $\boxed{8088}$

Solution. We look at the given equations for $n+1$ and n and subtract:

$$\begin{aligned} \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{n-1}} + \frac{1}{a_n} + \frac{n+1}{a_{n+1}} &= 1 \\ \frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_{n-1}} + \frac{n}{a_n} &= 1 \\ \frac{n+1}{a_{n+1}} - \frac{n-1}{a_n} &= 0. \end{aligned}$$

This produces the recursion $a_{n+1} = \frac{n+1}{n-1} \cdot a_n$ for $n \geq 2$. This is a telescoping product, and the closed form is $a_n = \frac{n(n-1)}{2} \cdot a_2 = 2n(n-1)$ for $n \geq 3$. Then, $a_{2023} - a_{2022} = 2 \cdot 2023 \cdot 2022 - 2 \cdot 2022 \cdot 2021 = 4 \cdot 2022 = \boxed{8088}$. $\square$

Problem 11. A sequence $a_0, a_1, \dots a_9, a_{10}$ of non-negative integers is called *slow* if, for all $0 \le k \le 9$, $|a_{k+1} - a_k| \le 1$. How many *slow* sequences are there such that $a_0 = 0$ and $a_{10} = 6$?

Answer. $\boxed{560}$

Solution. The difference between consecutive terms must be -1, 0, or 1, and the sum of all such differences must be $a_{10} - a_0 = 6$. We do casework on how many of these differences are 0.

Case 1: Four of the differences are 0. Then, the other six must be 1, and there are $\binom{10}{4} = 210$ ways to arrange them. Since all of the differences are non-negative in this case, all arrangements are valid, as all the terms will be non-negative. This gives us 210 sequences for this case.

Case 2: Two of the differences are 0. Then, there are seven 1s and one -1, with a total of $\frac{10!}{7!2!1!} = 360$ arrangements. In order to ensure all terms are non-negative, we must subtract all arrangements where the difference of -1 comes before any differences of 1. To do this, we can do casework on which difference -1 is. If it is the first difference, there are $\binom{9}{2} = 36$ ways to arrange the other differences. If it is the second, then the first must be a zero; $\binom{8}{1} = 8$ cases. If it is the third, both zeroes must come before, resulting in $\binom{7}{0} = 1$. We get $360 - 36 - 8 - 1 = 315$ sequences for this case.

Case 3: None of the differences are 0. Then, there are eight 1s, and two -1s, with a total of $\binom{10}{2} = 45$ arrangments. Like the previous case, we have to subtract the sequences where more -1s come before 1s. If the first difference is a -1, there are $\binom{9}{1} = 9$ arrangements. If the first difference is 1, then the next two must be -1, giving one sequence. We get $45 - 9 - 1 = 35$ sequences for this case.

Since no other number of differences can be 0, our answer is $210 + 315 + 35 = \boxed{560}$.

Remark. Many people neglected to read that all terms must be non-negative, which results in an answer of 615. □

Problem 12. Let $P(x)$ be a monic cubic polynomial with integer coefficients. Suppose it has roots a, b, and c that satisfy:

$$3 = \frac{(a+b)(b+c)(c+a)}{10} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Find the sum of all possible values of $|P(1)|$.

Answer. $\boxed{102}$

Solution. Let $P(x) = x^3 - qx^2 + rx - s = (x-a)(x-b)(x-c)$. Since $q = a+b+c$, $(a+b)(b+c)(c+a) = (q-c)(q-b)(q-a) = P(q) = q^3 - q^3 + qr - s = qr - s = 30$ by the first equation. The second equation says $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{ab+bc+ca}{abc} = \frac{r}{s} = 3$, so $r = 3s$.

Plugging in, we get $3qs - s = s(3q-1) = 30$. Since $P(x)$ has integer coefficients, we can simply check each factor of 30 (positive and negative), noting that $3q - 1$ must be 2 more than a multiple of 3. There are four valid triples (q, r, s), namely $(2, 18, 6)$, $(1, 45, 15)$, $(0, -90, -30)$, and $(-3, -9, -3)$. Then, $|P(1)| = |1 - q + r - s|$, so our answer is $|1-2+18-6| + |1-1+45-15| + |1-0-90+30| + |1+3-9+3| = \boxed{102}$. □

Problem 13. Let $f(x) = x^2 + nx + 6$, with n being a positive integer. Suppose there exist exactly 3 real solutions to the equation $f(f(f(x))) = f(x)$. Find n.

Answer. $\boxed{6}$

Solution. Notice that $f(x)$ is a quadratic, which means it has the useful property that $f(x) = f(-n-x)$, as its graph is symmetrical with respect to the line $x = -\frac{n}{2}$.

This means that for any solution t in the equation $f(f(f(x))) = f(x)$, $-n-t$ must also be a solution as:

$$f(t) = f(-n-t) \longrightarrow f(f(t)) = f(f(-n-t)) \longrightarrow f(f(f(t))) = f(f(f(-n-t)))$$

Thus, solutions come in pairs, except for the case where $t = -n-t$, or $t = -\frac{n}{2}$. And since we have exactly 3 real solutions, an odd number, this means that $t = -\frac{n}{2}$ must indeed be a solution.

From here the finish is simple. As $f(x) = x$ is an obvious solution to the equation, we simply plug in $x = t = -\frac{n}{2}$ to get:

$$\left(-\frac{n}{2}\right)^2 + n\left(-\frac{n}{2}\right) + 6 = -\frac{n}{2}$$

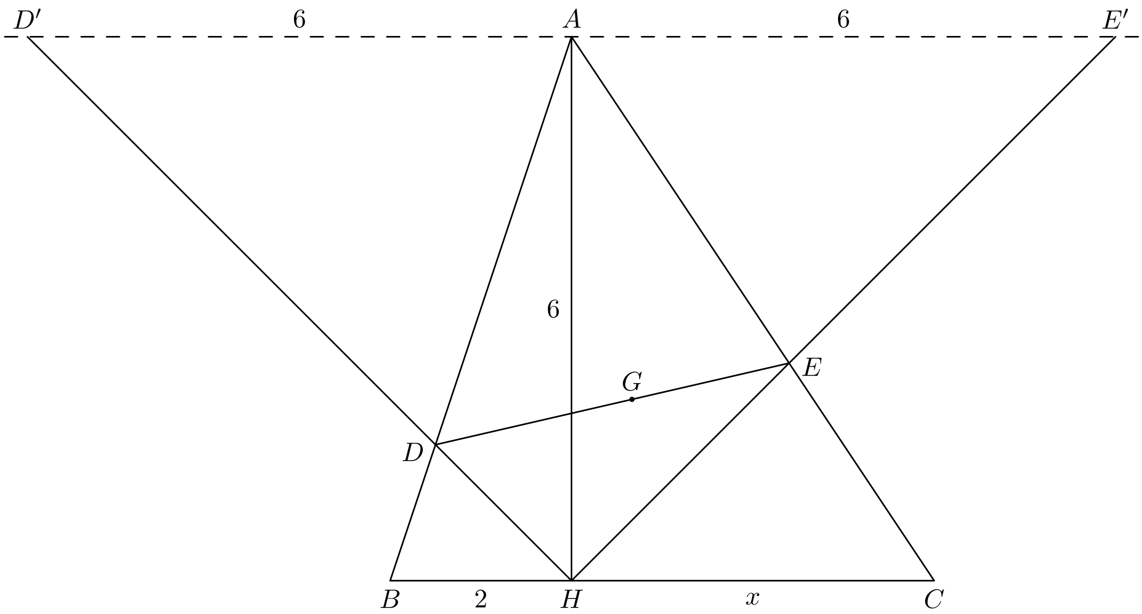
This is a quadratic with $n = 6$ and $n = -4$ as roots. As n has to be positive, we get the answer of $\boxed{6}$.

Remark. For a more rigorous finish, notice that as $t = -n-t$, we can remove one iteration of $f(x)$ to solve for: $f(f(t)) = t$, or $f(f(-\frac{n}{2})) = -\frac{n}{2}$. This is simply a quartic equation with two already known solutions $n = 6$ and $n = -4$, yielding the two non-integer solutions $n = 1 \pm \sqrt{29}$, which can also be rejected. □

Problem 14. Suppose we have $\triangle ABC$ with centroid G and altitude $AH = 6$ with point H on side $\overline{BC}$ such that $BH = 2$. Let point D be the intersection of the angle bisector of $\angle AHB$ and side $\overline{AB}$, and point E be the intersection of $\overline{AC}$ and the extension of $\overline{DG}$ through G. If $m\angle DHE = 90°$, find DE.

Answer. $\boxed{\dfrac{3\sqrt{178}}{10}}$

Solution. Let l be the line passing through A that is parallel to $\overline{BC}$. Extend $\overline{HD}$ past D and $\overline{HE}$ past E until they meet l at D' and E' respectively. Because $\angle DHE$ is right and $\overline{HD}$ bisects a right angle, $\triangle HD'E'$ is a right isosceles triangle and $AD' = AE' = AH = 6$. By parallel lines, $\triangle BDH \sim \triangle ADD'$ and $\triangle CEH \sim \triangle AEE'$ with ratios $2 : 6$ and $x : 6$ respectively.



We now define each point as a vector, with A being the origin. Then, $D = \frac{3}{4}B$ and $E = \frac{6}{x+6}C$ by the aforementioned similar triangle ratios. The centroid can be defined as the average of the vertices: $G = \frac{1}{3}B + \frac{1}{3}C$. Now, since D, G, and E are collinear, G can be expressed as a weighted average of D and E; that is, $G = r \cdot D + (1 - r) \cdot E$, meaning $\frac{1}{3}B + \frac{1}{3}C = r \cdot \frac{3}{4}B + (1 - r) \cdot \frac{6}{x+6}C$.

From this, $\frac{1}{3} = \frac{3}{4}r$ gives $r = \frac{4}{9}$, and $\frac{1}{3} = \frac{5}{9} \cdot \frac{6}{x+6}$ gives $x = 4$. Now, we drop the altitudes from D and E to $\overline{BC}$. Their lengths can be computed to be $6 \cdot \frac{1}{4} = \frac{3}{2}$ and $6 \cdot \frac{2}{5} = \frac{12}{5}$ respectively using the similar triangle ratios. Finally, by the Pythagorean Theorem, $DE = \sqrt{\left(\frac{12}{5} + \frac{3}{2}\right)^2 + \left(\frac{12}{5} - \frac{3}{2}\right)^2} = \boxed{\dfrac{3\sqrt{178}}{10}}$.

□

Problem 15. Suppose four distinct points are chosen from the vertices of a regular 24-gon to form a convex quadrilateral. How many different quadrilaterals are there with all angles less than $135°$? (Two quadrilaterals that are rotations of each other are considered different).

Answer. $\boxed{5346}$

Solution. Number the vertices of the 24-gon 0 through 23 in clockwise order. Then, fix A to be the vertex numbered 0, and define $a = B - A$, $b = C - B$, $c = D - C$, and $d = 24 - D$, where $ABCD$ is the chosen quadrilateral with vertices in clockwise order. In order for every angle to be less than $135°$, they must inscribe less than $270°$ of the circle, or $\frac{3}{4}$ of the 24 arcs that the 24-gon makes. This means $a+b, b+c, c+d, d+a < 18$. Coupled with the fact that $a + b + c + d = 24$, an equivalent condition is $6 < a + b, b + c < 18$.

We now generalize the answer based on what $a + b$ is. Let $a + b = n$. Then, there are $n - 1$ choices for B and $23 - n$ choices for D. However, since we cannot have $b + c \leq 6$ or $b + c \geq 18$, we must subtract the cases where B and D are too close together. Because $6 < n < 18$, any $b + d = k < 6$ is possible, and $k > 18$ is possible on the other side by symmetry. We now sum over $k = 2, 3, 4, 5, 6$ to get $1 + 2 + 3 + 4 + 5 = 15$, and we double this amount for $k = 22, 21, 20, 19, 18$. Our final count is

$$\sum_{n=7}^{17} [(n-1)(23-n) - 2 \cdot 15] = \sum_{n=6}^{16} [n(22-n) - 30]$$

$$= \sum_{n=1}^{16} \left[-n^2 + 22n - 30\right] - \sum_{n=1}^{5} \left[-n^2 + 22n - 30\right]$$

$$= f(16) - f(5)$$

where $f(n) = -\frac{n(n+1)(2n+1)}{6} + 11n(n+1) - 30n$. This gives $f(16) - f(5) = 891$. However, this is not the answer, because we fixed A in the beginning. We must multiply by 24 for the number of vertices of the 24-gon to choose from, and divide by 4 for the number of vertices of the quadrilateral we can pick to fix. Our final answer is $\frac{24}{4} \cdot 891 = \boxed{5346}$. $\square$