# NYCMT WHALE Solutions

NYCMT

March 2024

**Problem 1.** Let $n$ be the smallest positive perfect square such that $2024n$ is a perfect cube. Compute $\sqrt{n}$.

**Answer.** $\boxed{253}$

*Solution.* We note that $2024 = 2^3 \cdot 11 \cdot 13$. Because $n$ is a perfect square, its prime factorization consists only of even exponents. To minimize $n$, it should be in the form $2^{2a} \cdot 11^{2b} \cdot 13^{2c}$ for nonnegative integers $a$, $b$, and $c$. Any other prime factors are unnecessary.

Now, $2024n = 2^{3+2a} \cdot 11^{1+2b} \cdot 13^{1+2c}$ is a perfect cube, so its prime factorization consists only of exponents that are multiples of 3. This means that $3 \mid 3+2a, 1+2b, 1+2c$, and the smallest solutions are $a = 0$, $b = 1$, and $c = 1$.

Thus, the minimal $n$ is $11^2 \cdot 23^2$, and $\sqrt{n} = 11 \cdot 23 = \boxed{253}$. □

**Problem 2.** An Olympic-size swimming pool is 50 meters long and can hold up to 2500 cubic meters of water. Daniel is building a to-scale model that is 0.1 meters wide and 0.008 meters deep. How much water can Daniel's model hold, in cubic centimeters?

**Answer.** $\boxed{160}$

*Solution.* Daniel's scale model has a width to depth ratio of $0.1 : 0.008 = 25 : 2$ that must also be true in the Olympic-size swimming pool. Let its width be $25k$ and its height be $2k$. Then, $50 \cdot 25k \cdot 2k = 2500$ gives $k = 1$. This means that the pool's dimensions are 250 times the dimensions of the model, which has dimensions $0.2 \times 0.1 \times 0.008$ (in meters). Converting to centimeters, the answer is $20 \cdot 10 \cdot 0.8 = \boxed{160}$. $\square$

**Problem 3.** A string of letters is called *fruity* if there exists a pair of consecutive letters that are the same. For example, APPLE is fruity, but BANANA is not. The word GRAPEFRUIT is written on a chalkboard. Andrew erases a random letter each minute until the chalkboard is blank. The probability that the string of letters on the chalkboard is never fruity can be expressed as $\frac{m}{n}$ where $m$ and $n$ are relatively prime positive integers. Find $m + n$.

**Answer.** $\boxed{29}$

*Solution.* We use complementary counting, solving for the probability that the string of letters is, at some point, fruity. The only repeat letters in GRAPEFRUIT are the two Rs, so the only way for the string of letters to be fruity is if the A, P, E, and F are all erased before any of the Rs.

We can represent the order of letters erased as a random permutation of the letters in GRAPEFRUIT. Specifically, we want the probability that A, P, E, and F all come before both Rs. First, note that the four letters G, U, I, and T do not affect this probability; we can first construct the ordering of the other six letters, and insert these four anywhere to achieve every permutation. Then, we are looking at all permutations of AEFPRR, and it is clear that the two Rs must be the last two letters. Of the $\binom{6}{2} = 15$ ways of arranging the two Rs, only one satisfies this.

Our desired probability is then $1 - \frac{1}{15} = \frac{14}{15}$, and $m + n = 14 + 15 = \boxed{29}$.

$\square$

**Problem 4.** Find the smallest prime $p$ such that $p^2 + 2024$ has 27 divisors.

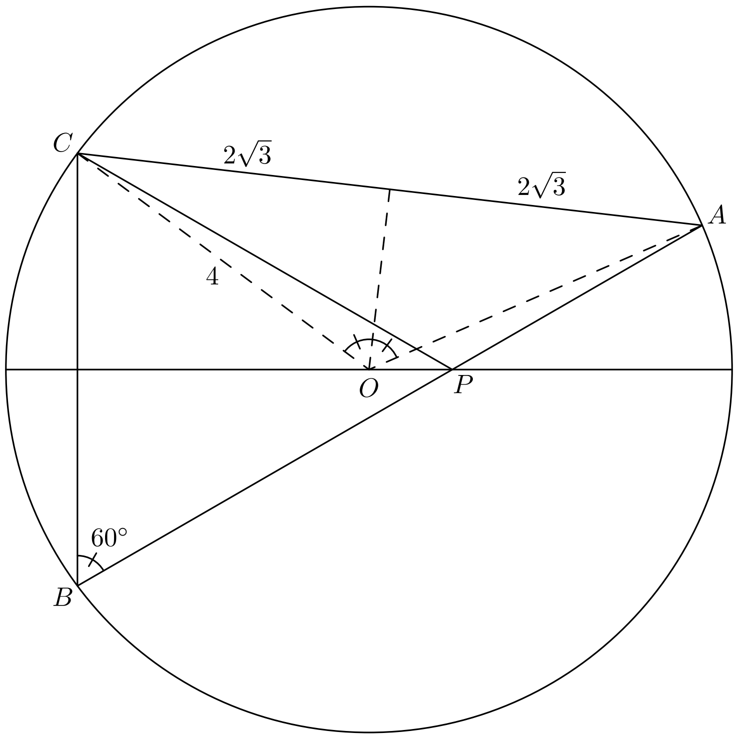
**Answer.** $\boxed{251}$

*Solution.* If a number has an odd number of divisors, then it must be a square. This means that we can let $p^2 + 2024 = n^2$ for some positive integer $n$. Then, $n^2 - p^2 = 2024 = 2^3 \cdot 11 \cdot 23 = (n-p)(n+p)$. These two factors $(n-p, n+p)$ must be the same parity, so the only possible choices are $(44, 46)$, $(22, 92)$, $(4, 506)$, and $(2, 1012)$. The first case gives $p = 1$, which is not prime. The second case gives $p = 35$, which is also not prime. The third gives $p = 251$, which is prime, and $n = 255$. Since $255^2 = 3^2 \cdot 5^2 \cdot 17^2$ does indeed have 27 factors, our answer is $\boxed{251}$. $\square$

**Problem 5.** Point $P$ lies on diameter $\overline{XY}$ of a circle with radius 4. Chord $\overline{AB}$ passes through $P$ and makes a $30^\circ$ degree angle with $\overline{XY}$. Let $C$ be the reflection of $B$ over $\overline{XY}$. If $\frac{AP}{PB} = \frac{2}{3}$, then the area of $\triangle ABC$ can be expressed as $\frac{a\sqrt{b}}{c}$, where $a$ and $c$ are relatively prime positive integers and $b$ is a squarefree integer. Find $a + b + c$.

**Answer.** $\boxed{202}$

*Solution.* By reflection, $m\angle BPC$ is twice the angle $\overline{PB}$ makes with the diameter, which is $2 \cdot 30^\circ = 60^\circ$. By symmetry, $\angle PBC \cong \angle PCB$, so $\triangle BPC$ is equilateral. Then, $m\angle ABC = 60^\circ$, so $\overline{AC}$ inscribes a $120^\circ$ arc. We can compute the length of $\overline{AC}$ by drawing the altitude from $O$, the center of the circle, and using $30-60-90$ triangle ratios to get that $AC = 2 \cdot 4\sin(60^\circ) = 4\sqrt{3}$.




Now, let $AP = 2x$ and $PB = 3x$, which gives $BC = 3x$ and $AB = 5x$. By Law of Cosines on $\triangle ABC$,

$$(3x)^2 + (5x)^2 - 2(3x)(5x)\cos(60^\circ) = (4\sqrt{3})^2$$

which gives $x^2 = \frac{48}{19}$. The area of $\triangle ABC$ is then

$$\frac{1}{2}(3x)(5x)\sin 60^\circ = \frac{15\sqrt{3}}{4}x^2 = \frac{180\sqrt{3}}{19}$$

by Law of Sines, and the desired sum is $a + b + c = 180 + 3 + 19 = \boxed{202}$. □

**Problem 6.** Find the unique integer $k$ such that the polynomial $x^3 - 14x^2 + 62x - k$ has three zeroes that are the side lengths of a right triangle.

**Answer.** $\boxed{84}$

*Solution.* Let the three zeroes of the polynomial be $r$, $s$, and $t$ such that $t$ is the largest. Then, it must be the length of the hypotenuse, and $r^2 + s^2 = t^2$ by the Pythagorean Theorem. Adding $t^2$ to both sides gives a symmetric expression on the left-hand side, which we can compute with Vieta's:

$$\begin{aligned} r^2 + s^2 + t^2 &= 2t^2 \\ (r+s+t)^2 - 2(rs+st+tr) &= 2t^2 \\ (14)^2 - 2(62) = 72 &= 2t^2 \\ t &= \pm 6. \end{aligned}$$

We reject the negative solution, as side lengths must be positive. Then, since 6 is a zero of the cubic, we can plug in:

$$(6)^3 - 14(6)^2 + 62(6) - k = 0$$

and $k = \boxed{84}$. □

**Problem 7.** Three standard six-sided dice are rolled, and let $A$ be their sum. Then, the die with the lowest number is re-rolled. (If there are multiple dice with the lowest number, only one of them is re-rolled.) Let $B$ be the sum of the dice after the re-roll. The probability that $B > A$ can be expressed as $\frac{n}{6^4}$. Find $n$.

**Answer.** $\boxed{855}$

*Solution.* We do casework on what the lowest number is after the initial rolls of the three dice, and sum the probabilities. Note that, for an integer $k \le 6$, there are $6 - k + 1$ integers $x$ that satisfy $k \le x \le 6$ and $6 - k$ integers $x$ that satisfy $k < x \le 6$.

This means that the probability that the lowest number is $k$ is

$$\frac{(6-k+1)^3 - (6-k)^3}{6^3}$$

as there are $(6 - k + 1)^3$ rolls in which the lowest number is at least $k$, and $(6 - k)^3$ rolls in which the lowest number is greater than $k$.

If the lowest number rolled is $k$, then there is a $\frac{6-k}{6}$ chance of rolling a greater number (and therefore greater sum) after the re-roll. We then want to compute

$$\begin{aligned}
n &= 6^4 \cdot \sum_{k=1}^{6} \frac{(6-k+1)^3 - (6-k)^3}{6^3} \cdot \frac{6-k}{6} \\
&= \sum_{k=1}^{6} \left((6-k+1)^3 - (6-k)^3\right) \cdot (6-k) \\
&= \sum_{k=0}^{5} \left((k+1)^3 - k^3\right) \cdot k \\
&= 5(6^3 - 5^3) + 4(5^3 - 4^3) + 3(4^3 - 3^3) + 2(3^3 - 2^3) + 1(2^3 - 1^3) \\
&= 5 \cdot 6^3 - \sum_{k=1}^{5} k^3 \\
&= 1080 - 225 = \boxed{855}.
\end{aligned}$$

$\square$

**Problem 8.** How many ways are there to color the faces of a cube one of five colors such that no two faces sharing an edge are the same color? Rotations are considered distinct.

**Answer.** $\boxed{780}$

*Solution.* We fix the colors of the top and bottom faces. If they are the same color, then there are 5 ways to color them, and the remaining four faces must be colored with the other four colors only. If they are different colors, then there are $5 \cdot 4 = 20$ ways to color them, and the remaining four faces must be colored with the other three colors only.

We define $f(n)$ as the number of ways to color the left, front, right, and back faces of a cube with $n$ colors while satisfying the condition. Then, our answer is $5f(4) + 20f(3)$.

Because these four faces form a loop around the cube, $f(n)$ is equivalent to the number of length 4 sequences of $n$ colors, where adjacent colors are different, and the first and last colors are different. We will use complementary counting, computing sequences with distinct adjacent colors, and then subtracting away sequences where the first and last colors are the same.

The invalid sequences are of the form $ABCA$, and there are $n(n-1)(n-2)$ ways to choose which colors. The total number of sequences is clearly $n(n-1)^3$, as there are $n$ ways to choose the first color and $n-1$ ways to choose each of the other three. So, $f(3) = 3 \cdot 2^3 - 3 \cdot 2 \cdot 1 = 24 - 6 = 18$, and $f(4) = 4 \cdot 3^3 - 4 \cdot 3 \cdot 2 = 108 - 24 = 84$.

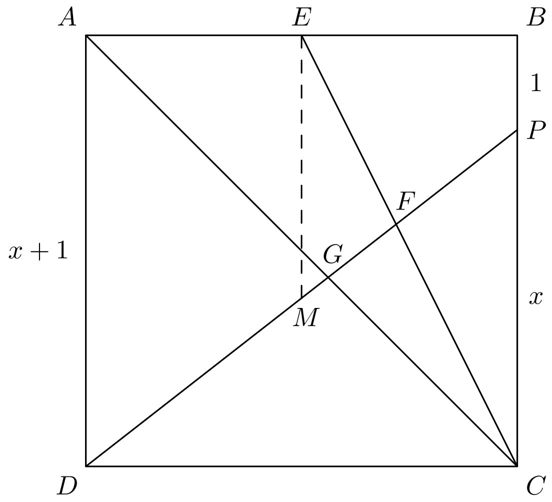
Our answer is then $5 \cdot 84 + 20 \cdot 18 = 420 + 360 = \boxed{780}$. □

*Remark.* Using this strategy allows for generalization to any number of colors, as $f(n) = n(n-1)^3 - n(n-1)(n-2)$ in general.

**Problem 9.** Square $ABCD$ has $E$ as the midpoint of $\overline{AB}$. Let $P$ be the point on $\overline{BC}$ such that the line $\overleftrightarrow{PD}$ intersects $\overline{EC}$ and $\overline{AC}$ at $F$ and $G$ respectively, and $\frac{AG}{GC} = \frac{CF}{FE}$. If $\frac{CP}{PB}$ can be expressed as $\frac{a+\sqrt{b}}{c}$, where $a$ and $c$ are relatively prime positive integers and $b$ is a squarefree integer, find $a + b + c$.

**Answer.** $\boxed{22}$

*Solution.* WLOG, let $BP = 1$ and $PC = x$, so $\frac{CP}{PB} = x$. Then, $AD = BC = x + 1$. By parallel lines, $\triangle AGD \sim \triangle CGP$ with ratio $\frac{AD}{CP} = \frac{x+1}{x} = \frac{AG}{GC}$.




Let $M$ be the midpoint of $\overline{PD}$. Then, $\overline{EH}$ is the midline of trapezoid $BPDA$, and has length $\frac{BP+AD}{2} = \frac{x+2}{2}$. By parallel lines, $\triangle CFP \sim \triangle EFM$ with ratio $\frac{CP}{EM} = \frac{x}{\frac{x+2}{2}} = \frac{2x}{x+2} = \frac{CF}{FE}$.

We are given that the two ratios $\frac{AG}{GC}$ and $\frac{CF}{FE}$ are equal, so we get $\frac{x+1}{x} = \frac{2x}{x+2}$ and $x^2 - 3x - 2 = 0$, which has positive solution $x = \frac{3+\sqrt{17}}{2}$. The desired sum is then $a + b + c = 3 + 17 + 2 = \boxed{22}$. $\square$

**Problem 10.** Consider a sequence of non-negative integers defined with $x_1, x_2 < 1000$, and $x_k = \min\{|x_i - x_j|, 0 < i < j < k\}$ for all integers $k \geq 3$. For example, $x_3 = |x_1 - x_2|$ and $x_4 = \min\{|x_1 - x_2|, |x_1 - x_3|, |x_2 - x_3|\}$. Find the greatest possible value of $x_{17}$ over all such sequences.

**Answer.** $\boxed{0}$

*Solution.* We claim that the sequence $x_3, x_4, \ldots$ is non-increasing. By the definition of $x_k$ for $k \geq 3$, there must exist $i < j < k$ such that $|x_i - x_j| = x_k$ is minimal over all $i$ and $j$. Since it is also true that $i < j < k + 1$, we know $x_{k+1} \leq x_k$ by the definition of $x_{k+1}$.

We can extend this non-increasing property to $x_1$ and $x_2$ WLOG. First, we can clearly assume $x_1 \geq x_2$ WLOG, as their absolute difference, equal to $x_3$, does not change. Now, if $x_2 \geq x_3$, we are done. If $x_2 < x_3$, we swap $x_2$ and $x_3$, using the fact that $x_1 - x_2 = x_3 \implies x_1 - x_3 = x_2$.

We move on to maximizing the value of $x_{17}$. It is clearly not optimal to have $x_k = x_{k+1}$, as $x_{k+2} = 0$, and this does not maximize terms past $x_{k+2}$, so we would like to avoid this for as long as possible. In order for $x_k \neq x_{k+1}$, the value of $x_{k+1}$ must be the minimum absolute difference involving $x_k$, as other absolute differences are already accounted for in $x_k$'s definition. In fact, because $x_n$ is non-increasing, this minimum absolute difference must equal $x_{k-1} - x_k$.

The recursion $x_{k+1} = x_{k-1} - x_k$ becomes more familiar if we reverse the sequence. Let $y_1 = x_{17}$, $y_2 = x_{16}$, and $y_k = x_{18-k}$ for $1 \leq k \leq 17$. Rearranging the recursion gives $y_{k+1} = y_k + y_{k-1}$, the Fibonacci recursion. From this, the answer is the maximum integer $A = x_{17} = y_1$ such that the sequence $y_n$ is non-decreasing, follows the Fibonacci recursion, and $y_{17} = x_1 < 1000$.

The smallest non-trivial setup $y_1 = y_2 = 1$ gives $y_{17} = 1597$, the 17th Fibonacci number, which is greater than 1000. As a result, $A = \boxed{0}$.

*Remark.* If we are trying to maximize $y_1$, it is clearly not optimal for $y_2$ to be greater than $y_1$. As a result, the optimal sequence is always the Fibonacci sequence, scaled by some integer. The maximum value of an arbitrary $x_n$ is then $\left\lfloor \frac{1000}{F_n} \right\rfloor$, where $F_n$ is the $n$th Fibonacci number. $\square$

**Problem 11.** In isosceles $\triangle ABC$ with $AB = AC = 10$, circles $\omega_1$ and $\omega_2$ are centered at $B$ and $C$ with radii 6 and 8, respectively. Point $G_1$ is on $\omega_1$ such that $AG_1 = \frac{32}{3}$ and $\overline{AG_1}$ intersects $\omega_1$ again at point $F_1$. Point $G_2$ is on $\omega_2$ such that $AG_2 = 8$ and $\overline{AG_2}$ intersects $\omega_2$ again at point $F_2$. If the circumradius of $\triangle AF_1F_2$ is 4, the positive difference between the maximum and minimum value of $G_1G_2^2$ can be expressed as $\frac{m}{n}$, where $m$ and $n$ are relatively prime positive integers.

**Answer.** $\boxed{563}$

*Solution.* First, we proceed with Power of a Point, noticing that we can compute the lengths of $AF_2$ and $AF_1$ easily given $AG_1$, $AG_2$, the radii of the circles and the length of $AB = AC$. Indeed, we have:

$$AF_1 \cdot AG_1 = (10-6)\cdot(10+6)$$

$$AF_2 \cdot AG_2 = (10-8)\cdot(10+8)$$

Plugging in appropriate values, we find that $AF_1 = 6$ and $AF_2 = \frac{9}{2}$.

Now, we wish to relate $F_1F_2$ to $G_1G_2$ in some kind of way, as we are only given information regarding $\triangle AF_1F_2$ and want to use that to get information about $G_1G_2$ Here, we can notice that $F_1F_2$ is parallel to $G_1G_2$ as $\frac{AF_1}{AG_1} = \frac{6}{\frac{32}{3}} = \frac{9}{16}$ and $\frac{AF_2}{AG_2} = \frac{\frac{9}{2}}{8} = \frac{9}{16}$.

Now let's use the information about $\triangle AF_1F_2$'s circumradius. We know that via the extended Law of Sines, $\frac{F_1F_2}{\sin A} = 2R = 8$, giving $\sin A = \frac{F_1F_2}{8}$. Now, because $F_1F_2$ and $G_1G_2$ are parallel, we know that $\triangle AF_1F_2 \sim AG_1G_2$ with similarity ratio $\frac{16}{9}$. This tells us that $G_1G_2 = \frac{16F_1F_2}{9} = \frac{128\sin A}{9}$.

Looking at $\triangle AG_1G_2$, we proceed with using the Law of Cosines, as we know the lengths $AG_1$, $AG_2$, and the other two variables are known in terms of angle $A$. Thus, we have:

$$\cos A = \frac{\left(\frac{32}{3}\right)^2 + 8^2 - \left(\frac{128\sin A}{9}\right)^2}{2\cdot\frac{32}{3}\cdot 8}$$

Setting $\sin A = \sqrt{1-\cos A^2}$, the equation becomes a simple quadratic where $\cos A = -\frac{1}{8}$ or $\cos A = \frac{31}{32}$. Thus there are only two possible values of angle $A$ which correspond to the minimum and maximum value of $G_1G_2^2$ respectively. Remembering $\sin^2 A = 1 - \cos^2 A$, we have:

$$\max\left(G_1G_2^2\right) - \min\left(G_1G_2\right) = \left(\frac{128}{9}\right)^2 \cdot \left(1 - \left(-\frac{1}{8}\right)^2\right) - \left(\frac{128}{9}\right)^2 \cdot \left(1 - \left(\frac{31}{32}\right)^2\right) = \frac{560}{3}.$$

Our desired sum is $560 + 3 = \boxed{563}$. □

**Problem 12.** Andrew loves painting whales. There are 100 blue whales floating in a circle. An integer $n$ is chosen with $2 \leq n \leq 50$. In a given move, Andrew chooses a set of $n$ consecutive whales that are floating adjacently, with the first and the last whales being blue, and paints the first and last whales white. Find the sum of all values of $n$ for which Andrew can paint all 100 whales white after 50 moves.

**Answer.** $\boxed{950}$

*Solution.* Number the whales from 0 to 99 in clockwise order, and let "whale $k$" refer to whale $m$, where $k \equiv m \pmod{100}$ and $0 \leq m \leq 99$. Because each move paints two whales that are separated by $n-1$, two whales numbered $x$ and $y$ can be painted white if both are blue and $x - y \equiv \pm(n-1) \pmod{100}$.

Let $(x, y)$ denote a move that paints whale $x$ and whale $y$ white. Then, WLOG, let the move that paints whale 0 white be $(0, n-1)$. Now, if we want to paint whale $2n-2$ white, there is only one possible move, namely $(2n-2, 3n-3)$, as whale $n-1$ is already painted white. We are forced to repeat this process, painting whales $(2k-2)(n-1)$ and $(2k-1)(n-1)$ on move $k$ until we run out of whales. There are then two cases to consider.

Case 1: There are an odd number of whales achieved by adding $n-1$ around the circle until we get back to whale 0. This is bad, because each move paints two whales at a time. Once we get to the last blue whale, its neighbors (separated by $(n-1)$) will already be painted. Then, it is clearly impossible to paint all 100 whales.

Case 2: There are an even number of whales achieved by adding $n-1$ around the circle until we get back to whale 0. In this case, repeating the process allows us to paint all whales achieved in this way. By Bézout's Lemma, all multiples of $\gcd(n-1, 100)$ are now painted white. Then, rotating this set of white whales to start at whale 1, 2, ... as necessary covers all whales.

It is now clear that we want to characterize when Case 2 occurs. Let $g = \gcd(n-1, 100)$. We want there to be an even number of multiples of $g$. The condition is then $2 \mid \frac{100}{g}$, or $g \mid 50$. In order for $\gcd(n-1, 100)$ to divide 50, $n-1$ cannot be a multiple of 4, as 50 is not divisible by 4. Any other value of $n$ works, as $g$ must be a factor of 100.

Our final answer is the sum of all $n$, minus all $n$ that are 1 more than a multiple of 4: $(2+3+\cdots+50)-(5+9+\cdots+49) = \frac{1}{2}\cdot 49\cdot 52 - \frac{1}{2}\cdot 12\cdot 54 = \boxed{950}$. $\square$