1. **[6]** What is the smallest possible sum of squares of four distinct integers?

Team Name: ______________ Answer: ______________

. . . .

2. **[6]** Find the smallest positive integer n such that n has 8 positive integer factors.

Team Name: ______________ Answer: ______________

. . . .

3. **[7]** Let a and b be positive integers such that the base 9 integer $n = \underline{ab}_9$ is equal to the base 5 integer $\underline{ba}_5$. Find the sum of all possible values of n (in base 10).

Team Name: ______________ Answer: ______________

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4. **[7]** Let a and b be positive integers with $a < b < 100$. When the rational number $\frac{a}{b}$ is reduced to lowest terms, the numerator and denominator sum to 10. Find the greatest possible value of a.

Team Name: ______________ Answer: ______________

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5. **[8]** Compute the sum of all real x such that $(\log_{10} x^4)^2 = (\log_{10} x)^6$.

Team Name: ______________ Answer: ______________

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6. **[8]** Rectangle $ABCD$ has $AB = 8$ and $AD = 12$. Let M be the midpoint of AD and N the midpoint of BC. Then, X is on AB such that $CX \perp BM$ and Y is on CD such that $AY \perp DN$. Compute the area of the quadrilateral bounded by lines AY, BM, CX, and DN.

Team Name: ______________ Answer: ______________

7. **[9]** The city is trying to light up a road that is 240 meters long by placing some number of streetlights along the road. Each end must have one streetlight, and all streetlights must be separated the same, integer number of meters. Find the sum of all possible numbers of streetlights the city could place.

Team Name: ____________ Answer: ____________

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8. **[9]** Quadrilateral $ABCD$ has $\angle BAD = \angle ADC = 90°$. Point E is drawn on line CD such that $\angle EBD = 90°$. Given that $EC = CD = 17$ and $AD = 15$, compute the largest possible area of quadrilateral $ABED$.

Team Name: ____________ Answer: ____________

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9. **[10]** Daniel and Aditya are playing five chess matches, where the players draw with probability $\frac{1}{2}$ and are otherwise equally likely to win. Find the probability that Aditya wins a majority of the matches that don't end in draws.

Team Name: ____________ Answer: ____________

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10. **[Up to 10]** You and the other NYCTC teams are competing in a game of Battleship. To play, submit a quadruple of real numbers (s_x, s_y, t_x, t_y), each of which is written as a decimal (that is, you should submit something like 1.434 and not an expression like $\frac{2+\pi e}{3!}$). This places a ship at (s_x, s_y) and shoots a torpedo at (t_x, t_y). You win

$$\frac{200}{20 + s_x^2 + s_y^2}$$

points, unless your ship gets sunk due to being within 1 unit of any torpedo (including your own), in which case you get 0 points.

Team Name: ____________ Answer: ____________

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11. **[11]** Find the sum of all prime numbers p such that $13p + 1$ is a perfect cube.

Team Name: ____________ Answer: ____________

12. **[11]** Triangle $\triangle ABC$ has side lengths $AB = 28$ and $AC = 36$. Point P is drawn such that $\triangle PBA \sim \triangle PAC$, and it's given that $AP = 21$. Then, points X and Y satisfy $ABP \sim \triangle AXB$ and $\triangle ACP \sim \triangle AYC$. What is the value of $AX \cdot AY$?

Team Name: ______________ Answer: ______________

. . . .

13. **[12]** Find the sum of y over all positive solutions (x, y) to the following system of equations:

$$x\lfloor y \rfloor = 20$$
$$y\lfloor x \rfloor = 23$$

As a reminder, $\lfloor r \rfloor$ is the greatest integer that is at most r.

Team Name: ______________ Answer: ______________

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14. **[12]** If a rectangular prism with integer dimensions has the same surface area and volume, what is the maximum possible value of its volume?

Team Name: ______________ Answer: ______________

. . . .

15. **[13]** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that satisfies the equation

$$f(x)f(y) = yf(x) + xf(2y)$$

for all real x and y. What is the maximum possible value of $f(17)$?

Team Name: ______________ Answer: ______________

. . . .

16. **[13]** Find the sum of all positive integers n less than 100 such that the divisors of n sum to twice a prime.

Team Name: ______________ Answer: ______________

17. **[14]** Penguino is at $(0,0)$. He wants to go to $(6,6)$, where his best friend Geont resides. Every minute, Penguino waddles one unit in a direction parallel to one of the axes, being careful to avoid $(3,4)$ and $(2,2)$, as the ice there is much too thin. If Penguino reaches Geont in 12 minutes, how many different paths could he have taken while avoiding the thin ice?

Team Name: _____________ Answer: _____________

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18. **[14]** Find the sum of all $x \in [0, 28\pi]$ such that

$$5\cot x + 5\tan x + 11 = 0.$$

Team Name: _____________ Answer: _____________

. . . .

19. **[15]** Cyclic quadrilateral $ABCD$ has circumcenter O. It is known that $AB = 3$, $CD = 5$, $\angle BOC = 73^\circ$, and $\angle AOD = 167^\circ$. Compute the area of the circumcircle of $ABCD$.

Team Name: _____________ Answer: _____________

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20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.

(1) Two players are, in turn, placing the first 9 positive integers on a 3×3 grid. The first player wins if the sum of the numbers in the top row is greater than the sum of the numbers in the left column. Then, with optimal play, the first player doesn't win.

(2) There exists exactly one function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x)f(y)f(z) - f(xyz) = xy + yz + zx + x + y + z$.

(3) Given $\triangle ABC$, point P inside the triangle is *jolly* if $\angle PAB = \angle PBC = \angle PCA = 30^\circ$. If $\triangle ABC$ is chosen such that a jolly point exists, it must be equilateral.

(4) Given an integer $b > 1$, the integer $n > 1$ is *b-based* if n is equal to the sum of the squares of its base b digits. Then, there exists a b-based number for each odd b.

(5) There exist infinitely many monic cubic polynomials with integer coefficients and roots r, s, and t such that $r + s + t = r^2 + s^2 + t^2 = r^3 + s^3 + t^3$.

(6) For a positive integer k, let $\varphi(k)$ be the number of positive integers at most k and relatively prime to k. Then,

$$\frac{n}{\varphi(n)} < 2023^{2023^{2023}}$$

is true for all positive integers n.

Team Name: _____________ Answer: _____________

21. **[16]** In front of Emperor Daniel, there lie 2023 piles of Kit Kats, where the kth pile contains k Kit Kats. One move consists of eating an equal number of Kit Kats from any subset of the piles. What is the least number of moves that Daniel needs to eat all the Kit Kats?

Team Name: _____________ Answer: _____________

. . . .

22. **[16]** A strictly increasing sequence of positive integers a_1, a_2, a_3, $\ldots$ is *Fibonacci-like* if, for each positive integer n,

$$a_{n+2} = a_{n+1} + a_n.$$

Compute the largest positive integer M for which there is a unique Fibonacci-like sequence satisfying $a_8 = M$.

Team Name: _____________ Answer: _____________

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23. **[17]** Let a, b, c, and x be real numbers such that

- $|x| \le 1$.
- for all real t satisfying $|t| \le 1$, $|at^2 + bt + c| \le 2023$.

As a, b, c, and x vary under these constraints, what is the largest possible value of $|2ax + b|$?

Team Name: _____________ Answer: _____________

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24. **[17]** In acute triangle $\triangle ABC$, the altitude from A intersects BC at D. The feet of the altitudes from D to AB and AC are P and Q respectively. If $BD = 8$, $CD = 9$, and $\sin A = \frac{3}{4}$, find the ratio of the area of $\triangle DPQ$ to the area of $\triangle ABC$.

Team Name: _____________ Answer: _____________

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25. **[18]** Find all positive integers n such that there exist exactly 2023 values of α in $(0, 90^\circ)$ satisfying

$$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \cdots + \sin(2n-1)\alpha = 0.$$

Team Name: _____________ Answer: _____________

26. **[18]** Let $\mathcal{P}$ be a right square pyramid with apex X and base $ABCD$. The altitude from X to the base has midpoint M. If the distance from M to the plane containing $\triangle ABX$ is $\sqrt{2}$ and the distance from M to line AX is $\sqrt{3}$, compute the volume of $\mathcal{P}$.

Team Name: _____________ Answer: _____________

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27. **[19]** Given a permutation $(a_1, a_2, \ldots, a_n)$ of n real numbers, an *inversion* is a pair (a_i, a_j) such that $a_i > a_j$ and $i < j$. How many permutations of the first 7 positive integers are there with exactly three inversions?

Team Name: _____________ Answer: _____________

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28. **[19]** An arithmetic sequence of positive integers with k terms and common difference 12 has the following property: the product of all the terms of this sequence is a divisor of a number of the form $n^2 + 1$ for some integer n. What is the largest possible value of k?

Team Name: _____________ Answer: _____________

. . . .

29. **[20]** In $\triangle ABC$, the angle bisector of $\angle BAC$ intersects BC at D. Let I be the incenter of $\triangle ABC$, O_1 the circumcenter of $\triangle ABD$, O_2 the circumcenter of $\triangle ACD$, and O the circumcenter of $\triangle ABC$. It is given that A, I, O_1, and O_2 are concyclic. If the circumradius of $\triangle ABC$ is 10, the length of OI is 3, and the area of $\triangle ABC$ is 115, compute the area of $\triangle AO_1O_2$.

Team Name: _____________ Answer: _____________

. . . .

30. **[20]** For a positive integer k, let $\varphi(k)$ be the number of positive integers at most k and relatively prime to k. Find the sum of the first five odd numbers n for which $\varphi(n)$ has the same number of divisors as n.

Team Name: _____________ Answer: _____________