NYCTC Fall 2023 Results and Solutions

NYCTC Writers

December 26, 2023

Contents

1 Summary and Results

The 2023 Fall NYCTC had a record 144 signups, partitioned into 41 teams. Overall, the contest was harder than anticipated, with the last six problems having a solve rate of 0%, and teams having only reached problem 15 at the halfway point. Some of the earlier problems in particular were harder than expected. Remarks on interesting solve rate statistics are scattered throughout the solutions.

1.1 Acknowledgements

The contest was written and managed by David Jiang, Jaemin Kim, Andrew Li, Mikayla Lin, Haokai Ma, Aditya Pahuja, Noam Pasman, Daniel Potievsky, and Calvin Zhang. Writers for each problem are credited in the solutions below.

Many thanks as well to Rishabh Das, Vidur Jasuja, and Jacob Paltrowitz for testsolving and providing feedback on fairly short notice.

1.2 Results

The race for top 3 was extremely close, all being within 7 points of each other. Second and third place were separated by only one point! Congratulations to the top three teams, who are enumerated below:

🥇 **1st Place.** Team Lehigh Mountain Water, consisting of Sophia Jin, Gabe Levin, James Papaelias, and Kyle Wu, with 208.75 points.

🥈 **2nd Place.** Team Passionfruit, consisting of Ian Buchanan, Anastasia Lee, Steven Lou, and Emma Yang, with 203.44 points.

🥉 **3rd Place.** Team Roggenrola, consisting of Tanvir Ahmed, Steven Breger, Corwin Eisenbeiss, and Jed Sloan, with 202.44 points.

A special congratulations goes to Lehigh Mountain Water for claiming their second NYCTC win. Full standings can be found at this link. Thanks to all participants for trying the contest! We hope you had fun, and we'll see you again in the spring.

2 Solutions

Problem 1. What is the smallest possible sum of squares of four distinct integers?

Proposed by Aditya Pahuja

The answer is $(-1)^2 + 0^2 + 1^2 + 2^2 = \boxed{6}$.

Problem 2. Find the smallest positive integer n such that n has 8 positive factors.

Proposed by Andrew Li

If n has 8 positive factors, then its prime factorization is either p^7, p^3q, or pqr for some primes p, q, r. In the first case, $p^7 \geq 128$; in the second, $p^3q \geq 2^3 \cdot 3 = 24$; in the third, $pqr \geq 2 \cdot 3 \cdot 5 = 30$. Thus, the answer is $\boxed{24}$.

Problem 3. Let a and b be positive integers such that the base 9 integer $n = \underline{ab}_9$ is equal to the base 5 integer $\underline{ba}_5$. Find the sum of all possible values of n (in base 10).

Proposed by Aditya Pahuja

The given equality can be represented as

$$9a + b = 5b + a \iff 4b = 8a \iff b = 2a.$$

Since $\underline{ba}_5$ is a base 5 integer, its digits are at most 4, so the only working pairs (a, b) are $(1, 2)$ and $(2, 4)$, giving $n = 11$ and $n = 22$ in base 10. Their sum is $\boxed{33}$.

Problem 4. Let a and b be positive integers with $a < b < 100$. When the rational number $\frac{a}{b}$ is reduced to lowest terms, the numerator and denominator sum to 10. Find the greatest possible value of a.

Proposed by Andrew Li

We can easily see that $\frac{a}{b}$ must be equal to either $\frac{1}{9}$ or $\frac{3}{7}$. The first case gives $b \leq 9 \cdot 11 = 99$ and $a \leq 11$, while the second case gives $b \leq 7 \cdot 14 = 98$ and $a \leq 42$. This means $(a, b) = (42, 98)$ achieves the maximal value of a, and the answer is $\boxed{42}$.

Problem 5. Compute the sum of all real x such that $(\log_{10} x^4)^2 = (\log_{10} x)^6$.

Proposed by Daniel Potievsky

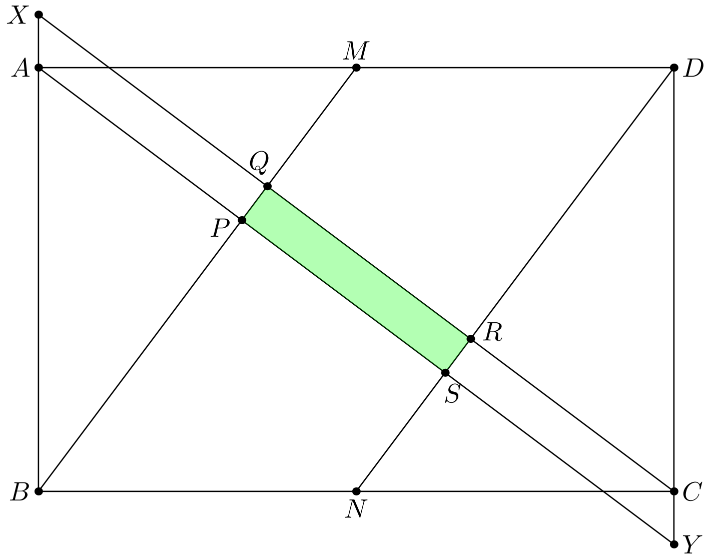
Let $y = \log_{10} x$. Then, the equation becomes

$$(4y)^2 = y^6,$$

which is equivalent to $y^6 - 16y^2 = y^2(y^4 - 16) = 0$. This implies that $y^2 = 0$ or $y^4 = 16$. The first case gives $y = 0$ and so $x = 1$, while the second case gives the solutions $y = 2$ and $y = -2$, corresponding to $x = 100$ and $x = \frac{1}{100}$. Thus, the answer is $100 + 1 + \frac{1}{100} = \boxed{\frac{10101}{100}}$.

Problem 6. Rectangle $ABCD$ has $AB = 8$ and $AD = 12$. Let M be the midpoint of AD and N the midpoint of BC. Then, X is on AB such that $CX \perp BM$ and Y is on CD such that $AY \perp DN$. Compute the area of the quadrilateral bounded by lines AY, BM, CX, and DN.

Proposed by Aditya Pahuja



Let AY intersect BM and DN at P and S respectively, and let CX intersect BM and DN at Q and R respectively. Then, we want the area of rectangle $PQRS$.

Observe that all the right triangles in this diagram are 3-4-5 triangles because they are all similar to $\triangle ABM$, which has side lengths 6, 8, and $\sqrt{6^2 + 8^2} = 10$. This means that

$$BQ = BC \cdot \frac{3}{5} = \frac{36}{5}, \quad MP = AM \cdot \frac{3}{5} = \frac{18}{5}.$$

We can then compute

$$PQ = BQ + MP - BM = \frac{36}{5} + \frac{18}{5} - 10 = \frac{4}{5}.$$

Then, we see that $\triangle APM \sim \triangle ASD$ with scale factor 2, meaning P is the midpoint of AS or $AP = PS$. Thus

$$PS = AP = AM \cdot \frac{4}{5} = \frac{24}{5}.$$

To finish, the area of $PQRS$ is

$$PQ \cdot PS = \frac{4}{5} \cdot \frac{24}{5} = \boxed{\frac{96}{25}}.$$

Problem 7. The city is trying to light up a road that is 240 meters long by placing some number of streetlights along the road. Each end must have one streetlight, and all streetlights must be separated by the same, integer number of meters. Find the sum of all possible numbers of streetlights the city could place.

Proposed by Andrew Li

The streetlights can be separated by m meters if and only if $\frac{240}{m}$ is an integer; that is, m must divide 240. In this case, we get a total of $\frac{240}{m}+1$ streetlights. Thus, we want the sum

$$\sum_{m|240} \frac{240}{m}+1,$$

taken over all divisors of 240. Since divisors of 240 come in pairs that look like $(m, \frac{240}{m})$, this rewrites as

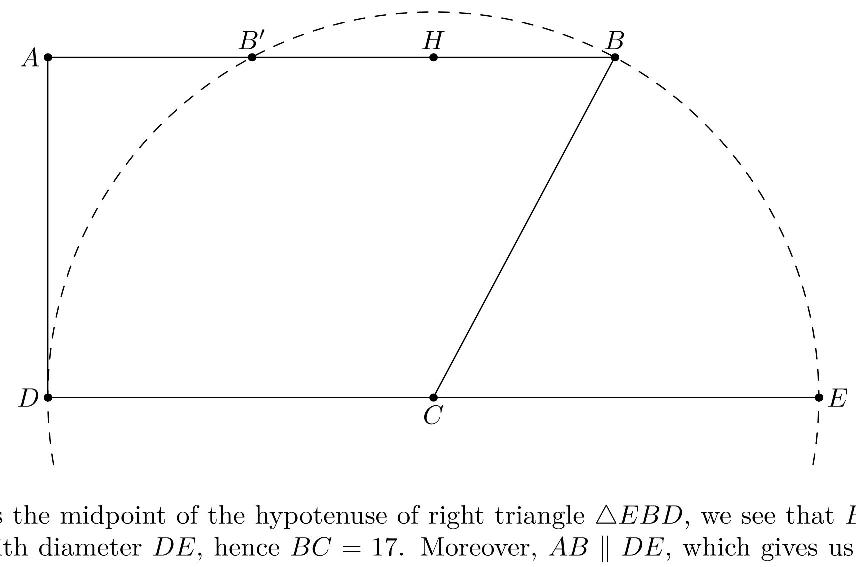
$$\sum_{m|240} m+1 = \sum_{m|240} m + \sum_{m|240} 1.$$

The first sum is the sum of the divisors of 240, while the second one is the number of divisors of 240. This evaluates to

$$(1+2+2^2+2^3+2^4)(1+3)(1+5)+(1+4)(1+1)(1+1) = 744+20 = \boxed{764}.$$

Problem 8. Quadrilateral $ABCD$ has $\angle BAD = \angle ADC = 90^\circ$. Point E is drawn on CD such that $\angle EBD = 90^\circ$. Given that $EC = CD = 17$ and $AD = 15$, compute the largest possible area of quadrilateral $ABED$.

Proposed by Aditya Pahuja



Since C is the midpoint of the hypotenuse of right triangle $\triangle EBD$, we see that B must lie on the circle with diameter DE, hence $BC = 17$. Moreover, $AB \parallel DE$, which gives us two possible locations for B. We want $ABED$ to have as large an area as possible, thus we pick the point further from A, so that base AB is as long as possible.

Now, it remains to do the computation. Let H be the the foot of the altitude from C to AB. Then $BH = \sqrt{17^2-15^2} = 8$, $AH = CD = 17$, and $DE = 34$, which means the area of $ABED$ is

$$\frac{AD(DE+AB)}{2} = \frac{15(34+17+8)}{2} = \frac{15\cdot 59}{2} = \boxed{\frac{885}{2}}.$$

Remark 8.1. This problem had a solve rate of 56.41%, which is much lower than expected.

Problem 9. Daniel and Aditya are playing five chess matches, where the players draw with probability $\frac{1}{2}$ and are otherwise equally likely to win. Find the probability that Aditya wins a majority of the matches that don't end in draws.

Proposed by Daniel Potievsky

The answer is $\boxed{\frac{193}{512}}$. We present two solutions.

Solution 1. Let p be the probability that the number of wins equals the number of losses. Since Aditya and Daniel are equally likely to win any given game, the answer is $\frac{1-p}{2}$; that is, it's half the probability that someone wins more games than the other.

We compute p by considering each possible number of draws individually.

- Suppose there is one draw. The probability of this is happening $\binom{5}{1} \cdot \left(\frac{1}{2}\right)^5$, as the probability of any given game being a draw is $\frac{1}{2}$, and we have $\binom{5}{1}$ choices for which game is a draw here. Of the four non-draw games, the probability that Aditya and Daniel win the same number of games is $\binom{4}{2} \cdot \left(\frac{1}{2}\right)^4$, which gives an overall probability of
$$\binom{5}{1} \cdot \left(\frac{1}{2}\right)^5 \cdot \binom{4}{2} \cdot \left(\frac{1}{2}\right)^4 = \frac{15}{256}.$$
- Suppose there are three draws. The probability of this is happening $\binom{5}{3} \cdot \left(\frac{1}{2}\right)^5$, with the same logic as above. Of the two non-draw games, the probability that Aditya and Daniel win the same number of games is $\binom{2}{1} \cdot \left(\frac{1}{2}\right)^2$, which gives an overall probability of
$$\binom{5}{3} \cdot \left(\frac{1}{2}\right)^5 \cdot \binom{2}{1} \cdot \left(\frac{1}{2}\right)^2 = \frac{5}{32}.$$
- If there are five draws, then the outcome of the matches is fixed at each player getting 0 wins, which contributes $\frac{1}{32}$ to p.
- If there is an even number of draws, then one player is guaranteed to have more wins than the other, meaning this case contributes nothing to p.

Thus,
$$\frac{15}{256} + \frac{5}{32} + \frac{1}{32} = \frac{63}{256} \implies \frac{1-p}{2} = \frac{1}{2} \cdot \frac{256-63}{256} = \boxed{\frac{193}{512}}.$$

Solution 2. We again compute p and then find $\frac{1-p}{2}$, as in Solution 1. Consider the polynomial
$$\left(\frac{x^{-1}}{4} + \frac{x^0}{2} + \frac{x^1}{4}\right)^5.$$

When we compute any term of this polynomial, we pick one term from each of the 5 factors and then multiply the terms, which models our scenario exactly, as we can envision picking the x^1 term to be Aditya winning, picking the x^{-1} term to be Daniel winning, and picking the x^0 term to be a draw; the product of encodes the probability (via the coefficient) and how many more games Aditya wins than Daniel (via the exponent).

For example, the sequence $WLWDD$, where W means Aditya winning, L means Daniel winning, and D means a draw, has probability $\frac{1}{256}$ and one net win for Aditya. Accordingly, it corresponds to the term
$$\frac{x^1}{4} \cdot \frac{x^{-1}}{4} \cdot \frac{x^1}{4} \cdot \frac{x^0}{2} \cdot \frac{x^0}{2} = \frac{x^1}{256}.$$

Thus, p is the constant term of this polynomial. Multiplying by x^5, this constant term becomes the x^5 coefficient of
$$\frac{1}{4^5}(x^2+2x+1)^5 = \left(\frac{x+1}{2}\right)^{10}.$$

The binomial theorem tells us that the x^5 coefficient is $\frac{1}{2^{10}} \cdot \binom{10}{5} = \frac{63}{256} = p$, and then $\frac{1-p}{2} = \boxed{\frac{193}{512}}$.

Problem 10. You and the other NYCTC teams are competing in a game of Battleship. To play, submit a quadruple of real numbers (s_x, s_y, t_x, t_y), each of which is written as a decimal (that is, you should submit something like 1.434 and not an expression like $\frac{2+\pi e}{3!}$). This places a ship at (s_x, s_y) and shoots a torpedo at (t_x, t_y). You win

$$\frac{200}{20 + s_x^2 + s_y^2}$$

points, unless your ship gets sunk due to being within 1 unit of any torpedo (including your own), in which case you get 0 points.

Proposed by Calvin Zhang

The highest score was 7.34, achieved by team BJs's submission of $(-1, 2.5, -1, 1.5)$. Twelve teams had their ships struck by torpedoes. The full list of submissions is below:

Team Name	**Submission**	**Score**
Lehigh Mountain Water	$(-4.3, 1.9, 4.3, 1.9)$	4.75
SFBA A	$(-7, -3, 3, 4)$	2.56
Passionfruit	$(-4, 3, 2, 2)$	4.44
Roggenrola	$(0, 5.0001, 1, 1)$	4.44
dancing mashed penguin potato cow	$(5, 5, 1.41, 1.41)$	2.86
Gonk	$(-1.001, -1.001, 1, 1)$	0
Joshua's Team	$(-2, 6, -2, 4)$	3.33
Alligatorz Rule!1!	$(-4, 3, 0, 0)$	4.44
🥝	$(7.9, 1.4, 6.9, 4.2)$	2.37
i believe i can fly	$(-8.3, 2.7, -3, -5)$	2.08
M(oguss)y Shrill Bump	$(3.9, 1.69, 0, 2.69)$	0
I wanna go to NYCMT.	$(0, 0, 1, 1)$	0
actually ashley	$(-11.4, -8.7, 3.5, 4.5)$	0.89
Dad left to go get milk	$(9, 2, 2, 2)$	1.9
Dad came back with the Milk	$(3, -3, 7, 7)$	5.26
Dad never came back with the Milk	$(-11.5, 1, 0, 0)$	1.31
duck duck duck goose	$(-2, 2, 10, 10)$	7.14
Damaged	$(-3.9, 3.9, .9, -.9)$	3.97
Potatoes	$(4, 3, 2, 1)$	0
FRAUDULENT HEDGEHOGS DERIVE	$(1, 1, 8, 8)$	0
Gucci Gang	$(7, -3, 0, 0)$	2.56
Olive Eulers	$(0, 5.5, -3, -4)$	3.98
Shrimp	$(7, 1, 6, 9)$	2.86
We don't have a Good Team Name and We don't want to Think One	$(-2.5, 2.5, 2, 2)$	6.15
老干妈 Furrets	$(-2.1, -4.9, 1, 2)$	0
#N/A	$(2.12, -2.12, 0, 0)$	6.9
BJs	$(-1, 2.5, -1, -1.5)$	7.34
Flying Pigs	$(2.5, 2.7, 0.99, 0.99)$	0
In god we trust	$(2, -4, -2, -2)$	5
It was me barry	$(-8.412, 2.32, 3, 4)$	2.08
u snooze u lose	$(-5, 5, 1, 2)$	2.86
Don't Forget +C	$(0, 4, 6, 9)$	5.56
jackiee	$(2.5, 3.1, 1, 2.1)$	0
0 on IMO	$(4, 4, 9, 9)$	0
Stuy - 1	$(-2.1, -2.1, 3, 3)$	0
liam x bryce canon	$(8, 2, 3, 6)$	2.27
No Solution	$(-4, -1, 1, 0)$	5.41
Avocadoes 🥑	$(0, 0, 19, 15)$	0
!rand team_name	$(-2, -4, 0, 0)$	0
Haokai	$(-8, -1, 0, 0)$	2.35

Problem 11. Find the sum of all prime numbers p such that $13p + 1$ is a perfect cube.

Proposed by Daniel Potievsky

Let $13p + 1 = k^3$, for some integer k. Then

$$13p = k^3 - 1 = (k-1)(k^2 + k + 1).$$

Since $13p$ is a product of two primes, $k - 1$ and $k^2 + k + 1$ must be prime unless one of them is 1. However, $k > 2$ because $k^3 = 13p + 1 > 13 > 2^3$, so both factors must be prime.

In particular, one of the factors is 13. If $k - 1 = 13$, then $k = 14$ and then $p = k^2 + k + 1 = 211$, which we can check is prime. If $k^2 + k + 1 = 13$, then $k = 3$ and $p = k - 1 = 2$, which is prime.

Thus $p = 2$ and $p = 211$ are the two solutions, giving an answer of $2 + 211 = \boxed{213}$.

Problem 12. Triangle $\triangle ABC$ has side lengths $AB = 28$ and $AC = 36$. Point P is drawn such that $\triangle PBA \sim \triangle PAC$, and it's given that $AP = 21$. Then, points X and Y satisfy $\triangle ABP \sim \triangle AXB$ and $\triangle ACP \sim \triangle AYC$. What is the value of $AX \cdot AY$?

Proposed by Aditya Pahuja

There are several ways to draw this diagram, but the computation all boils down to the following: Observe that

$$\frac{AX}{AB} = \frac{AB}{AP}, \quad \frac{AY}{AC} = \frac{AC}{AP}.$$

Multiplying the equations together gives

$$\frac{AX \cdot AY}{AB \cdot AC} = \frac{AB \cdot AC}{AP^2} \iff AX \cdot AY = \frac{AB^2 \cdot AC^2}{AP^2} = \left(\frac{28 \cdot 36}{21}\right)^2 = \boxed{2304}.$$

> **Remark 12.1.** As you may have noticed, $\triangle PBA \sim \triangle PAC$ is entirely unnecessary.

Problem 13. Find the sum of y over all positive solutions (x, y) to the following system of equations:

$$x\lfloor y \rfloor = 20$$
$$y\lfloor x \rfloor = 23$$

As a reminder, $\lfloor r \rfloor$ is the greatest integer that is at most r.

Proposed by Jaemin Kim

We simply do casework on the value of $\lfloor y \rfloor$.

$\lfloor y \rfloor$	x	$\lfloor x \rfloor$	$\frac{23}{\lfloor x \rfloor} = y$
1	20	20	23/20
2	10	10	23/10
3	20/3	6	23/6
4	5	5	23/5
5	4	4	23/4
6	20/3	3	23/3
7	20/7	2	23/2
8	5/2	2	23/2
9	20/9	2	23/2
10	2	2	23/2
11	20/11	1	23
12	20/12	1	23
13	20/13	1	23
14	20/14	1	23
15	20/15	1	23
16	20/16	1	23
17	20/17	1	23
18	20/18	1	23
19	20/19	1	23
20	1	1	23

(This table looks longer than it really is; the arithmetic goes very quickly, especially in the last ten rows.) A value of $\lfloor y \rfloor$ generates a solution if and only if the rightmost column is consistent with the leftmost column; thus, only the first five rows give valid solutions. This means that the answer is

$$\frac{23}{4}+\frac{23}{5}+\frac{23}{6}+\frac{23}{10}+\frac{23}{20}=\boxed{\frac{529}{30}}.$$

Problem 14. If a rectangular prism with integer dimensions has the same surface area and volume, what is the maximum possible value of its volume?

Proposed by Aditya Pahuja

Let the prism's dimensions be $a \le b \le c$. Then

$$2(ab+bc+ca)=abc \iff \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{2}.$$

Since $\frac{1}{b}$ and $\frac{1}{c}$ are at most $\frac{1}{a}$, we must have

$$\frac{3}{a}\ge\frac{1}{2},$$

so a is at most 6 (and it's obviously at least 3). We can just manually check each possible value of a now.

- If $a=3$, then $\frac{1}{b}+\frac{1}{c}=\frac{1}{6}$, so

  $$bc=6b+6c \iff (b-6)(c-6)=36.$$

  Then the ordered pair (b,c) must be one of $(7,42)$, $(8,24)$, $(9,18)$, $(10,15)$, or $(12,12)$. Of these, $3\cdot 7\cdot 42=882$ gives the biggest volume.

- If $a=4$, then similar algebra leads to

  $$(b-4)(c-4)=16,$$

  so (b,c) is one of $(5,20)$, $(6,12)$, or $(8,8)$, of which $4\cdot 5\cdot 20=400$ gives the biggest volume.

- If $a=5$, then $3bc=10b+10c$, so $9bc=30b+30c$ and then

  $$(3b-10)(3c-10)=100.$$

  Both factors are two more than a multiple of 3, which forces $3b-10=5$ and $3c-10=20$ or $3b-10=2$ and $3c-10=50$. Respectively, these give $(b,c)=(5,10)$ and $(b,c)=(4,20)$, but the latter violates our $a\le b\le c$ assumption (and indeed, it has been counted in the previous case); so, the largest volume in this case is $5\cdot 5\cdot 10=250$.

- Finally, if $a=6$, then b and c must both be 6, as $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$ would necessarily be less than $\frac{1}{2}$ otherwise. This gives a volume of $6^3=216$.

The largest volume across all the cases is $\boxed{882}$.

Problem 15. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function that satisfies the equation

$$f(x)f(y) = yf(x) + xf(2y)$$

for all real x and y. What is the maximum possible value of $f(17)$?

Proposed by Aditya Pahuja

Clearly, $f(x) = 0$ for all x is a valid function f, so assume that f is not the zero function.

Claim 15.1 — $f(u) = 0$ if and only if $u = 0$.

Proof. First, we have

$$f(0)^2 = 0$$

by setting $x = y = 0$, so $f(0) = 0$.

Conversely, suppose there is a nonzero u such that $f(u) = 0$. Then

$$0 = f(u)f(y) = yf(u) + uf(2y) = uf(2y).$$

Since we are assuming that f is not the zero function, we can choose y such that $f(2y) \neq 0$, which forces $u = 0$. □

Now, letting $x = 2y \neq 0$, we have

$$f(2y)f(y) = yf(2y) + 2yf(2y) \iff f(y) = 3y$$

for all nonzero y (although if $y = 0$ then $f(y) = 0 = 3 \cdot 0$ anyway).

Thus, the only two functions that work are the zero function and the function $f(x) = 3x$, giving a maximum value of $\boxed{51}$ for $f(17)$.

Problem 16. Find the sum of all positive integers n less than 100 such that the divisors of n sum to twice a prime.

Proposed by Aditya Pahuja

Claim 16.1 — If the divisors of n sum to twice a prime, then n is prime.

Proof. Suppose that n has prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, and let $\sigma(n)$ be the sum of the divisors of n. Then

$$\sigma(n) = \sigma(p_1^{e_1})\sigma(p_2^{e_2}) \cdots \sigma(p_k^{e_k}).$$

Each of these factors is more than two (since the smallest possible value is $2 + 1 = 3$), so $k = 1$.

Now, $n = p^e$ for some prime p, so its divisors sum to

$$1 + p + p^2 + \cdots + p^e = 2q$$

for some prime q. Clearly $p \neq 2$, as the left-hand side would be odd otherwise. Then, e also needs to be odd for the same reason. This means that $\sigma(p^e)$ factors as

$$2q = (p^{(e+1)/2} + 1)(1 + p + p^2 + \cdots + p^{(e-1)/2}),$$

and as long as $e > 1$, both of these factors exceed 2, which is a contradiction. In other words, $e = 1$, so n is prime. □

Now we can manually check all the primes p less than 100 for whether $p + 1$ is twice a prime; we find that $p \in \{3, 5, 13, 37, 61, 73\}$, so the sum of the working p is $\boxed{192}$.

Problem 17. Penguino is at $(0, 0)$. He wants to go to $(6, 6)$, where his best friend Geont resides. Every minute, Penguino waddles one unit in a direction parallel to one of the axes, being careful to avoid $(3, 4)$ and $(2, 2)$, as the ice there is much too thin. If Penguino reaches Geont in 12 minutes, how many different paths could he have taken while avoiding the thin ice?

Proposed by Aditya Pahuja and Daniel Potievsky

Since Penguino reaches Geont in $6 + 6 = 12$ minutes, he must only move in the positive x and y directions. We can find that

(1) the number of paths to Geont without any other constraints is just $\binom{12}{6}$.

(2) the number of paths to Geont passing through $(2, 2)$ is $\binom{4}{2} \cdot \binom{8}{4}$.

(3) the number of paths to Geont through $(3, 4)$ is $\binom{7}{3} \cdot \binom{5}{3}$.

(4) the number of paths to Geont through both points is $\binom{4}{2} \cdot \binom{3}{1} \cdot \binom{5}{3}$.

We want to count the number of paths that are not type (2) or (3). This is done by subtracting off the number of type (2) and type (3) paths, then adding back the number of type (4) paths to account for overcounting.

$$\binom{12}{6} - \binom{4}{2} \cdot \binom{8}{4} - \binom{7}{3} \cdot \binom{5}{3} + \binom{4}{2} \cdot \binom{3}{1} \cdot \binom{5}{3} = 924 - 420 - 350 + 180 = \boxed{334}.$$

Remark 17.1. This problem had a staggering 66.67% solve rate!

Problem 18. Find the sum of all $x \in [0, 28\pi]$ such that

$$5 \cot x + 5 \tan x + 11 = 0.$$

Proposed by Daniel Potievsky

The given equation is a quadratic in $\tan x$, namely

$$5 \tan^2 x + 11 \tan x + 5 = 0.$$

Let α and β be the two possible values of $\tan x$, noting that $\alpha\beta = 1$ and that they are negative. If $\alpha = \tan(\theta)$ with $\frac{\pi}{2} < \theta < \pi$, then $\beta = \cot(\theta) = \tan(\frac{3\pi}{2} - \theta)$; importantly, both θ and $\frac{3\pi}{2} - \theta$ are in the interval $[0, \pi]$. Thus, the two possible values of x within $[0, \pi]$ sum to $\frac{3\pi}{2}$.

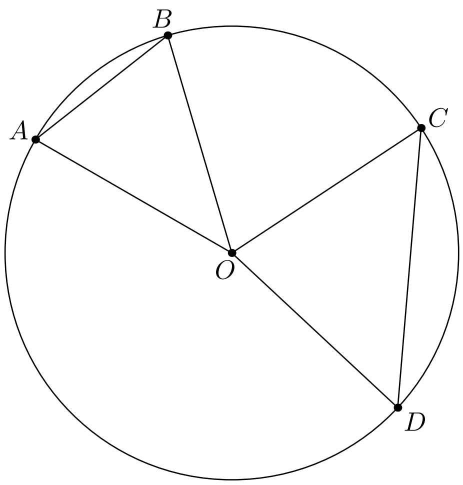
Now, $\tan x$ and $\cot x$ have periods of π, so the full curve of solutions x is given by $\theta + \pi k$ and $\frac{3\pi}{2} - \theta + \pi k$ for integers k. When constrained between 0 and 28π, the sum of the solutions turns out as

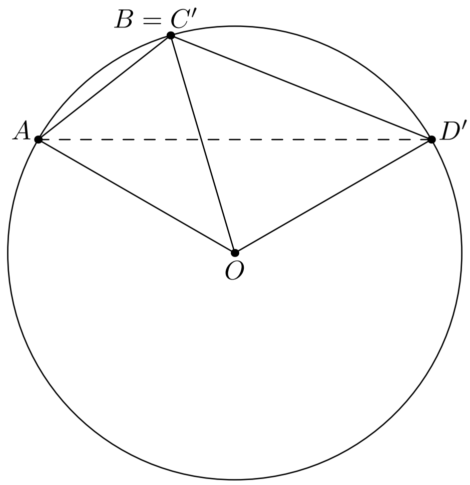
$$\sum_{k=0}^{27} \frac{3\pi}{2} + 2\pi k = 42\pi + \frac{27 \cdot 28}{2} \cdot 2\pi = \boxed{798\pi}.$$

Remark 18.1. In contrast with the previous problem, only 12.82% of teams solved this one.

Problem 19. Cyclic quadrilateral $ABCD$ has circumcenter O. It is known that $AB = 3$, $CD = 5$, $\angle BOC = 73°$, and $\angle AOD = 167°$. Compute the area of the circumcircle of $ABCD$.

Proposed by Aditya Pahuja





The main observation in this problem is that $\angle AOD + \angle BOC = 240°$. This motivates rotating $\triangle COD$ about O until the image of C is B, as shown in the diagram. Now, $\angle AOD' = 120°$, and we can find that $\angle ABD' = 120°$ as well. Since $AB = 3$ and $BD' = 5$, we can use the law of cosines to find

$$3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cdot \cos(120°) = 49 = (AD')^2.$$

Then, law of sines (or simply drawing the altitude from O to AD') allows us to compute

$$AO = \frac{AD'}{\sqrt{3}} = \frac{7}{\sqrt{3}},$$

which in turn means that the area of the circumcircle is

$$\left(\frac{7}{\sqrt{3}}\right)^2 \pi = \boxed{\frac{49\pi}{3}}.$$

Problem 20. Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ *points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.*

(1) Two players are, in turn, placing the first 9 positive integers on a 3×3 grid. The first player wins if the sum of the numbers in the top row is greater than the sum of the numbers in the left column. Then, with optimal play, the first player doesn't win.

(2) There exists exactly one function $f\colon \mathbb{R} \to \mathbb{R}$ such that $f(x)f(y)f(z) - f(xyz) = xy + yz + zx + x + y + z$.

(3) Given $\triangle ABC$, point P inside the triangle is *jolly* if $\angle PAB = \angle PBC = \angle PCA = 30^\circ$. If $\triangle ABC$ is chosen such that a jolly point exists, it must be equilateral.

(4) Given an integer $b > 1$, the integer $n > 1$ is b*-based* if n is equal to the sum of the squares of its base b digits. Then, there exists a b-based number for each odd b.

(5) There exist infinitely many monic cubic polynomials with integer coefficients and roots r, s, and t such that $r + s + t = r^2 + s^2 + t^2 = r^3 + s^3 + t^3$.

(6) For a positive integer k, let $\varphi(k)$ be the number of positive integers at most k and relatively prime to k. Then,

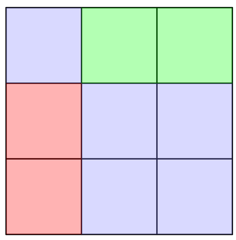
$$\frac{n}{\varphi(n)} < 2023^{2023^{2023}}$$

is true for all positive integers n.

Proposed by Aditya Pahuja and Daniel Potievsky

The answer is $\boxed{\text{FTTTTF}}$. Eight teams got a score higher than 1, with two teams, Passionfruit and BJs, getting the maximum possible score of 28.

(1) It turns out that player 1 has a winning strategy. Color the grid as follows.



Then, player 1 wants the sum of the green cells to be larger than the sum of the red cells; numbers placed in any of the blue cells don't contribute either way.

Suppose player 1 starts by placing 9 in a green cell. Then, player 2 must place 1 in the other green cell, since they need to block player 1 from placing a 7 or an 8 there.

Now, all player 1 needs to do is guarantee that the red cells sum to at most 9. A strategy to do so is as follows:

- Player 1 places 8 in a blue cell.
- Whenever player 2 places x, player 1 places $9 - x$ in a cell of the same color as x.

This works because the remaining numbers (after placing 8) come in pairs summing to 9 (namely $(2, 7)$, $(3, 6)$, and $(4, 5)$), so we are done.

(2) We show that $f(x) = x + 1$ is the only function that works; it is easy to see that $x + 1$ satisfies the equation.

First, $(x, y, z) = (0, 0, 0)$ gives
$$f(0)^3 - f(0) = 0,$$
so $f(0)$ is one of -1, 0, or 1. Then setting y and z to be 0 gives
$$f(x) \cdot f(0)^2 - f(0) = x.$$
If $f(0) = 0$, then we find that $x = 0$, which doesn't make sense because the equation is true for all real x. Thus $f(0)$ is 1 or -1. In either case, because $f(0)^2 = 1$, we have
$$f(x) = x + f(0),$$
and we can manually verify that $f(x) = x - 1$ doesn't satisfy the original equation, so we're done.

(3) Using the trigonometric form of Ceva's theorem, we find that
$$\frac{\sin(\angle PAB)}{\sin(\angle PAC)} \cdot \frac{\sin(\angle PCA)}{\sin(\angle PCB)} \cdot \frac{\sin(\angle PBC)}{\sin(\angle PBA)} = 1,$$
or equivalently,
$$\sin(\angle PAC)\sin(\angle PCB)\sin(\angle PBA) = \sin(30^\circ)^3 = \frac{1}{8}.$$
We will now show that
$$\sin(\angle PAC)\sin(\angle PCB)\sin(\angle PBA) \le \frac{1}{8},$$
with equality if and only if the three angles are equal to 30°.

Let $x = \angle PAC$, $y = \angle PCB$, and $z = \angle PBA$, so $x + y + z = 90^\circ$. Construct a triangle $\triangle XYZ$ with angles $\angle X = 2x$, $\angle Y = 2y$, and $\angle Z = 2z$. Then, let the bisector of $\angle X$ hit YZ at D and let H be the point on XD such that $YH \perp XD$. We see that
$$\sin x = \frac{HY}{XY} \le \frac{DY}{XY} = \frac{DZ}{XZ} = \frac{DX + DZ}{XY + XZ} = \frac{YZ}{XY + XZ}$$
by the angle bisector theorem, as well as the analogous expressions by symmetry. Note that equality only occurs when $HY = DY$, i.e. the angle bisector is perpendicular to YZ and $XY = XZ$. Thus
$$\begin{aligned}\sin x \sin y \sin z &\le \frac{XY}{YZ + ZX} \cdot \frac{YZ}{ZX + XY} \cdot \frac{ZX}{XY + YZ} \\ &\le \frac{XY \cdot YZ \cdot ZX}{2\sqrt{YZ \cdot ZX} \cdot 2\sqrt{YZ \cdot ZX} \cdot 2\sqrt{YZ \cdot ZX}} \\ &= \frac{1}{8}\end{aligned}$$
which follows from the AM-GM inequality. In order for equality to occur, we must have $XY = YZ = ZX$ and thus $x = y = z$, which implies that $\triangle ABC$ is equilateral.

Alternatively, one can observe that $\sin x + \sin y + \sin z \le \frac{3}{2}$ by Jensen's inequality and apply AM-GM from there.

(4) Let $b = 2k + 1$ for a positive integer k. Then, the number $bk + (k + 1)$ is b-based, as the squares of its digits in base b sum to
$$k^2 + (k + 1)^2 = 2k^2 + 2k + 1 = k(2k + 1) + (k + 1) = bk + (k + 1).$$

(5) Let $a = r + s + t$, $b = rs + st + tr$, and $c = rst$. Then, we can compute

$$a^2 - 2b = r^2 + s^2 + t^2 = a \implies b = \frac{a^2 - a}{2}$$

and

$$a(a - b) = (r + s + t)(r^2 + s^2 + t^2 - rs - st - tr) = r^3 + s^3 + t^3 - 3abc = a - 3c$$

which implies that

$$c = \frac{a - a^2 + ab}{3} = \frac{a^3 - 3a^2 + 2a}{6}.$$

Now, if a is any multiple of 6, the polynomial

$$x^3 - ax^2 + bx - c$$

will have the desired property, which gives an infinite set of working polynomials (actually, any integer a results in integer coefficients).

(6) Let n have prime factorization $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. Then,

$$\frac{n}{\varphi(n)} = \prod_{i=1}^{k} \frac{p_i^{e_i}}{(p_i - 1)p_i^{e_i - 1}} = \prod_{i=1}^{k} \frac{p_i}{p_i - 1}.$$

We show that this product diverges. Observe that

$$\frac{p}{p-1} = \frac{1}{1 - \frac{1}{p}} = 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots ,$$

so we can write the infinite product

$$\prod_{i=1}^{\infty} \frac{p_i}{p_i - 1} = \prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \frac{1}{p_i^3} + \cdots\right).$$

Multiplying this out would look like choosing one term from each factor of this expression and then multiplying the choices together; in fact, the fundamental theorem of arithmetic guarantees that every single positive integer appears in the denominator of exactly one term generated by this process. Thus,

$$\prod_{i=1}^{\infty} \left(1 + \frac{1}{p_i} + \frac{1}{p_i^2} + \frac{1}{p_i^3} + \cdots\right) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

The divergence of this sum is well-known, but here is a short proof for completeness: partition the natural numbers into sets of the form $\{2^k + 1, 2^k + 2, \ldots, 2^{k+1}\}$, so that

$$\frac{1}{2^k + 1} + \frac{1}{2^k + 2} + \cdots + \frac{1}{2^{k+1}} \geq \frac{2^k}{2^{k+1}} = \frac{1}{2}.$$

Since there are infinitely many such sets, we are adding $\frac{1}{2}$ infinitely many times, hence the sum diverges.

Problem 21. In front of Emperor Daniel, there lie 2023 piles of Kit Kats, where the kth pile contains k Kit Kats. One move consists of eating an equal number of Kit Kats from any subset of the piles. What is the least number of moves that Daniel needs to eat all the Kit Kats?

Proposed by Daniel Potievsky

Suppose that Daniel eats k_i Kit Kats from each pile that he eats from on his ith move. Then, after m moves, he can have eaten at most 2^m different numbers of Kit Kats from any pile, since the number of Kit Kats eaten is the sum of the elements of some subset of $\{k_1, k_2, \ldots, k_m\}$. This means that Daniel needs at least 11 moves to eat all the Kit Kats.

He can eat all the Kit Kats 11 moves by eating 2^{i-1} Kit Kats on move i from each pile whose ith digit in binary from the right is 1. For example, for $23 = 10111_2$, he will eat 1 on move 1, 2 on move 2, 4 on move 3, and 16 on move 5. Thus, the answer is $\boxed{11}$.

Remark 21.1. This problem had a surprisingly high solve rate of 30.77%.

Problem 22. A strictly increasing sequence of positive integers a_1, a_2, a_3, ...is *Fibonacci-like* if, for each positive integer n,

$$a_{n+2} = a_{n+1} + a_n.$$

Compute the largest positive integer M for which there is a unique Fibonacci-like sequence satisfying $a_8 = M$.

Proposed by Aditya Pahuja

Let $a_1 = x$ and $a_2 = y$. Then, we can compute $a_8 = 8x + 13y$. We therefore want

$$8x + 13y = M$$

to have a unique solution in positive integers with $x < y$.

If (u, v) satisfies the equation, then the full curve of integer solutions is given by $(u + 13k, v - 8k)$ for all integers k because x and y are fixed modulo 13 and modulo 8 respectively. Therefore, we just need to make sure that $(u + 13, v - 8)$ and $(u - 13, v + 8)$ both violate the "positive integers with $x < y$" constraint in some way to force (u, v) to be unique.

If $v \leq 8$, then $u < 8$, which gives an optimal pair $(u, v) = (7, 8)$ and a maximum of $8 \cdot 7 + 13 \cdot 8 = 160$.

On the other hand, if $v > 8$, we can allow u to be at most 13. Since $u + 13$ and $v - 8$ will then be positive integers, we need $u + 13 \geq v - 8$, or equivalently $v \leq u + 21$. This gives a maximum of $8 \cdot 13 + 13 \cdot 34 = 546$ when $(u, v) = (13, 34)$.

Since $\boxed{546}$ is the larger maximum of the two cases, it is the answer.

Remark 22.1. Some common pitfalls in this problem were forgetting that the sequence is strictly increasing (which gives an answer of 208) and forgetting that the sequence can't contain 0 (which, in conjunction with the previous assumption, gives an answer of 187).

Problem 23. Let a, b, c, and x be real numbers such that

- $|x| \le 1$.
- for all real t satisfying $|t| \le 1$, $|at^2 + bt + c| \le 2023$.

As a, b, c, and x vary under these constraints, what is the largest possible value of $|2ax + b|$?

Proposed by Daniel Potievsky

Since $2ax + b$ is linear in x, it is maximized at either $x = 1$ or $x = -1$, depending on whether a is positive or negative. Without loss of generality, assume $a > 0$, and then set $x = 1$ since that's optimal.

Let $f(y) = ay^2 + by + c$. Then,

$$\frac{3}{2}f(1) - 2f(0) + \frac{1}{2}f(-1) = \frac{3(a+b+c) - 4c + (a-b+c)}{2} = 2a + b = 2ax + b,$$

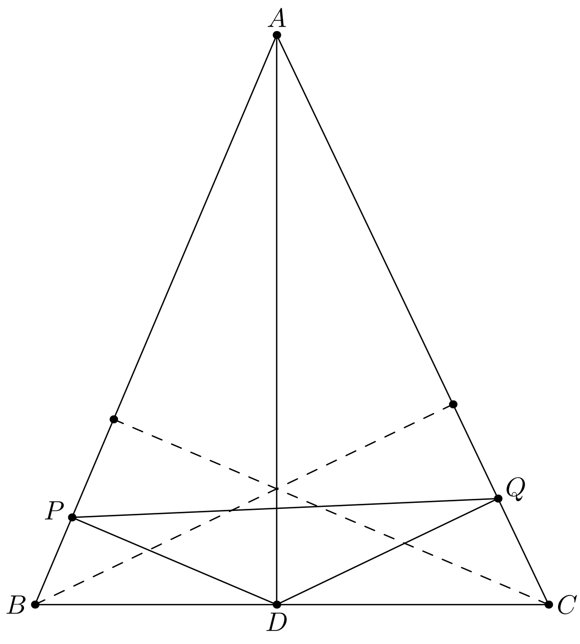
so by the triangle inequality,

$$|2ax + b| = |2a + b| \le \left|\frac{3}{2}f(1)\right| + |-2f(0)| + \left|\frac{1}{2}f(-1)\right| \le \frac{3}{2} \cdot 2023 + 2 \cdot 2023 + \frac{1}{2} \cdot 2023 = 8092.$$

This maximal value is achieved with $(a, b, c, x) = (4046, 0, -2023, 1)$, so $\boxed{8092}$ is indeed the answer.

Problem 24. In acute triangle $\triangle ABC$, the altitude from A intersects BC at D. The feet of the altitudes from D to AB and AC are P and Q respectively. If $BD = 8$, $CD = 9$, and $\sin A = \frac{3}{4}$, find the ratio of the area of $\triangle DPQ$ to the area of $\triangle ABC$.

Proposed by Aditya Pahuja



The answer is $\boxed{\frac{81}{578}}$. We present two solutions.

Solution 1. Let h_B be the length of the altitude from B and h_C the length of the altitude from C. Then,

$$DP = \frac{8h_C}{17}, \quad DQ = \frac{9h_B}{17}.$$

This lets us write the ratio of the areas as

$$\begin{aligned}\frac{\frac{1}{2}DP \cdot DQ \cdot \sin(\angle PDQ)}{\frac{1}{2}AB \cdot AC \cdot \sin(\angle BAC)} &= \frac{8h_C \cdot 9h_B \cdot \sin(180^\circ - \angle BAC)}{17^2 \cdot AB \cdot AC \cdot \sin(\angle BAC)} \\ &= \frac{72 \cdot AC \sin(\angle BAC) \cdot AB \sin(\angle BAC)}{289 \cdot AB \cdot AC} \\ &= \frac{72 \cdot \frac{3}{4} \cdot \frac{3}{4}}{289} = \boxed{\frac{81}{578}}.\end{aligned}$$

Solution 2 (by Sophia Jin). As in Solution 1, we want to find $\frac{DP \cdot DQ}{AB \cdot AC}$. Let $\alpha = \angle BAD$ and $\beta = \angle CAD$. Then,

$$DP = AD \sin\alpha, \quad DQ = AD \sin\beta, \quad AB = \frac{AD}{\cos\alpha}, \quad AC = \frac{AD}{\cos\beta}.$$

This means we need to find

$$\frac{AD \sin\alpha \cdot AD \sin\beta}{\frac{AD}{\cos\alpha} \cdot \frac{AD}{\cos\beta}} = \sin\alpha \sin\beta \cos\alpha \cos\beta.$$

Now, observe that

$$\begin{aligned}\frac{\tan\alpha}{\tan\beta} = \frac{\sin\alpha\cos\beta}{\cos\alpha\sin\beta} &= \frac{BD/AD}{CD/AD} = \frac{8}{9} \\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta &= \frac{3}{4}.\end{aligned}$$

This is a system of linear equations in $\sin\alpha\cos\beta$ and $\cos\alpha\sin\beta$, so we can find that $\sin\alpha\cos\beta = \frac{6}{17}$ and $\cos\alpha\sin\beta = \frac{27}{68}$. The answer is then their product, which is $\boxed{\frac{81}{578}}$.

Problem 25. Find all positive integers n such that there exist exactly 2023 values of α in $(0, 90^\circ)$ satisfying

$$\sin\alpha + \sin 3\alpha + \sin 5\alpha + \cdots + \sin(2n-1)\alpha = 0.$$

Proposed by Daniel Potievsky

The answer is $\boxed{n = 4047,\ n = 4048}$. We present two solutions.

Solution 1. Multiply the equation by $2\sin\alpha$. Then, using a product-to-sum identity,

$$2\sin\alpha\sin(2k-1)\alpha = \cos(2k-2)\alpha - \cos 2k\alpha.$$

Thus the equation becomes

$$\cos 0 - \cos 2\alpha + \cos 2\alpha - \cos 4\alpha + \cdots + \cos(2n-2)\alpha - \cos 2n\alpha = 0 \iff \cos 2n\alpha = 1.$$

This implies that $2n\alpha = 2\pi k$ for some integer k, so $\alpha = \frac{\pi k}{n}$. Since α is acute, $0 < \frac{k}{n} < \frac{1}{2}$. There are $\left\lfloor \frac{n-1}{2} \right\rfloor$ possible values of k satisfying this, so we need $\left\lfloor \frac{n-1}{2} \right\rfloor = 2023$, which implies that $\boxed{n = 4047 \text{ or } n = 4048}$.

Solution 2. Let $\sin\alpha = \frac{\omega - \frac{1}{\omega}}{2i}$ where $\omega = \cos\alpha + i\sin\alpha$. Then

$$\omega + \omega^3 + \cdots + \omega^{2n-1} = \frac{1}{\omega} + \frac{1}{\omega^3} + \cdots + \frac{1}{\omega^{2n-1}}.$$

This is equivalent to

$$\omega^{2n} + \omega^{2n+2} + \cdots + \omega^{4n-2} = \omega^{2n-2} + \omega^{2n-4} + \cdots + 1,$$

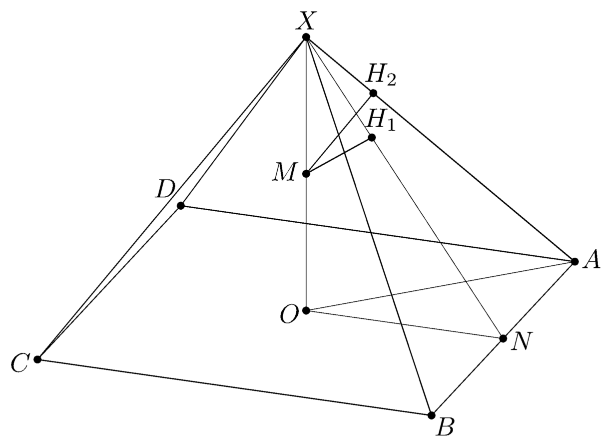
which rearranges as

$$(\omega^{2n} - 1)(\omega^{2n-2} + \omega^{2n-4} + \cdots + 1) = \frac{(\omega^{2n}-1)^2}{\omega^2 - 1} = 0.$$

This then shows that $\omega^{2n} = 1$, so $\alpha = \frac{\pi k}{n}$ for an integer k, and we proceed as in Solution 1.

Problem 26. Let $\mathcal{P}$ be a right square pyramid with apex X and base $ABCD$. The altitude from X to the base has midpoint M. If the distance from M to the plane containing $\triangle ABX$ is $\sqrt{2}$ and the distance from M to line AX is $\sqrt{3}$, compute the volume of $\mathcal{P}$.

Proposed by Daniel Potievsky



Let N be the midpoint of AB and let O be the center of $ABCD$, and define H_1 and H_2 to be the feet of M on the plane and line respectively. Also, let $OX = h$ and $AN = x$.

First, from $\triangle XNO \sim \triangle XMH_1$ and $\triangle XAO \sim \triangle XMH_2$, we get

$$\frac{h/2}{XN} = \frac{\sqrt{2}}{x}, \quad \frac{h/2}{XA} = \frac{\sqrt{3}}{x\sqrt{2}}.$$

The Pythagorean theorem says that $XN = \sqrt{x^2 + h^2}$ and $XA = \sqrt{2x^2 + h^2}$, so, squaring the similar triangle ratios, we have the system

$$\frac{h^2}{x^2 + h^2} = \frac{8}{x^2}$$
$$\frac{h^2}{2x^2 + h^2} = \frac{12}{2x^2}.$$

We can divide the two equations to get $6(2x^2 + h^2) = 8(x^2 + h^2)$, which implies that $h = x\sqrt{2}$. Thus

$$h = 2MX = 2 \cdot MH_2 \cdot \sqrt{2} = 2\sqrt{6}$$

and $x = 2\sqrt{3}$. The volume turns out to be

$$\frac{(2x)^2 \cdot h}{3} = \frac{4x^3\sqrt{2}}{3} = \boxed{32\sqrt{6}}.$$

Problem 27. Given a permutation $(a_1, a_2, \ldots, a_n)$ of n real numbers, an *inversion* is a pair (a_i, a_j) such that $a_i > a_j$ and $i < j$. How many permutations of the first 7 positive integers are there with exactly three inversions?

Proposed by Aditya Pahuja

For a nonnegative integer $r \le n$, let $a_{n,r}$ be the number of permutations of the first n positive integers with exactly r inversions.

Claim 27.1 — We can compute $a_{n+1,r}$ as

$$a_{n+1,r} = a_{n,r} + a_{n+1,r-1}$$

whenever $n \ge r \ge 1$.

Proof. We split into two cases. The first case is where $n+1$ is the last element of the permutation, in which case it contributes no inversions, and so we have $a_{n,r}$ ways to permute the rest.

The second case is where $n+1$ is not the last element. Let S be the set of r-inversion permutations satisfying this, and let T be the set of all $(n+1)$-element permutations with $r-1$ inversions. Then, consider the function $\phi\colon S \to T$ that swaps $n+1$ with the element to its right (for example $\phi(3,2,1) = (2,3,1)$). Each element of S gets mapped to a unique element of T by this function, thus ϕ is injective. Moreover, since $n > r-1$, it is not possible for the first element of a permutation in T to be an inversion, since otherwise (a_1, a_k) would be an inversion for $2 \le k \le n+1$; thus each element of T is in the image of S. In other words, ϕ is a bijection between the two sets, so $|S| = |T| = a_{n+1,r-1}$.

Adding the two cases gives the claimed recursion. □

Obviously, $a_{n,0} = 1$. We can compute $a_{n,1}$ as

$$a_{n,1} = a_{n-1,1} + a_{n,0} = a_{n-1,1} + 1,$$

and since $a_{2,1} = 1$, $a_{n,1} = n-1$. Then

$$a_{n,2} = a_{n-1,2} + a_{n,1} = a_{n-1,2} + n - 1$$

for $n > 2$. Since $a_{2,2} = 1$, this means $a_{n,2}$ is equal to

$$2 + 3 + 4 + \cdots + n - 1 = \binom{n}{2} - 1.$$

Finally, since $a_{3,3} = 1$, we find that for $n > 3$,

$$a_{n,3} = 1 + \binom{4}{2} - 1 + \binom{5}{2} - 1 + \cdots + \binom{n}{2} - 1 = \binom{n+1}{3} - n$$

via the hockey stick identity.

Plugging in $n = 8$ gives $\binom{8}{3} - 7 = \boxed{49}$ as the answer.

Problem 28. An arithmetic sequence of positive integers with k terms and common difference 12 has the following property: the product of all the terms of this sequence is a divisor of a number of the form n^2+1 for some integer n. What is the largest possible value of k?

Proposed by Daniel Potievsky

Claim 28.1 — If a positive integer m divides a number of the form n^2+1 where n is an integer, then all odd prime factors of m are 1 more than a multiple of 4.

Proof. We have
$$n^2 \equiv -1 \pmod{m}, \quad n^4 \equiv 1 \pmod{m}.$$
This means 4 is the smallest positive integer d for which $n^d \equiv 1 \pmod{m}$. Then, I claim that if $n^e \equiv 1 \pmod{m}$, e is a multiple of d. This follows from writing $e = dq + r$ for $0 \le r < d$ and observing that
$$n^e = (n^d)^q \cdot n^r \equiv n^r \equiv 1 \pmod{4},$$
which forces $r = 0$ as we would otherwise have contradicted the fact that $e = d$ is minimal.

Since $n^{p-1} \equiv 1 \pmod{p}$ for any p dividing n by Fermat's little theorem, $d \mid p-1$, i.e. $4 \mid p-1$. □

Now, we can always avoid having a multiple of 3 in the arithmetic sequence by letting the first term a be a non-multiple of 3. The next largest prime that isn't one more than a multiple of 4 is 7, and in fact this gives us an upper bound of $\boxed{k=6}$, since one of a, $a+12$, ..., $a+12\cdot 6$ is guaranteed to be a multiple of 7.

It remains to show that there exists a six-term sequence satisfying the constraints. Observe that the arithmetic sequence 5, 17, 29, 41, 53, 65 has terms whose prime factors are all $1 \pmod{4}$. Rather than just finding a corresponding n, we will prove the following generalization for entertainment value.

Lemma 28.2. If $m = p_1^{e_1} p_2^{e_2} \cdots p_j^{e_j}$, where the p_i are primes that are $1 \pmod{4}$, then there exists an integer n such that $m \mid n^2+1$.

Proof. First, we will show by induction on e that the congruence
$$n^2 \equiv -1 \pmod{p^e}$$
has a solution n for all $1 \pmod{4}$ primes p and all integers e.

For the base case of $e=1$, recall that by Wilson's theorem, $(p-1)! \equiv -1 \pmod{p}$. We can do some algebraic manipulation to construct a working n here:
$$\begin{aligned}
-1 &\equiv \left(1\cdot 2 \cdots \frac{p-1}{2}\right)\left(\frac{p+1}{2}\cdot\frac{p+3}{2}\cdots(p-1)\right) \\
&\equiv \left(1\cdot 2 \cdots \frac{p-3}{2}\cdot\frac{p-1}{2}\right)\left(\frac{-(p-1)}{2}\cdot\frac{-(p-3)}{2}\cdots(-2)\cdot(-1)\right) \\
&= \left(1\cdot 2 \cdots \frac{p-3}{2}\cdot\frac{p-1}{2}\right)^2\cdot(-1)^{(p-1)/2} \\
&= \left(1\cdot 2 \cdots \frac{p-3}{2}\cdot\frac{p-1}{2}\right)^2 \pmod{p}
\end{aligned}$$
where the last line is because $\frac{p-1}{2}$ is even.

Now, assume that the result holds for some e; that is, some r satisfies $r^2 \equiv -1 \pmod{p^e}$, so $r^2 = p^e d - 1$. Working modulo p^{e+1}, consider
$$\begin{aligned}
(p^e q + r)^2 &= p^{2e}q^2 + 2qrp^e + r^2 \\
&\equiv 2qrp^e + p^e d - 1 \\
&= (2qr+d)p^e - 1 \pmod{p^{e+1}}
\end{aligned}$$

for some integer q. In order for this square to be $1 \pmod{p^{e+1}}$, we just need p to divide $2qr + d$, which we can do by setting $q \equiv \frac{-d}{2r} \pmod{p}$, noting that $r \not\equiv 0 \pmod{p}$ because $0^2 \not\equiv -1 \pmod{p}$. This completes the induction.

Therefore, for each i, we have guaranteed the existence of an n_i such that

$$n_i^2 \equiv -1 \pmod{p_i^{e_i}}.$$

By the Chinese remainder theorem, we can then be sure that a solution to $n^2 \equiv -1 \pmod{m}$ exists by forcing $n \equiv n_i \pmod{p_i^{e_i}}$ for all i. □

Remark 28.3. If you were interested in just finding a working n in the example above, write each term as a sum of two squares. Then, the identity $(a^2+b^2)(c^2+d^2) = (ac-bd)^2+(ad+bc)^2$ lets us collapse the product into a sum of two squares, such as

$$17547^2 + 6346^2.$$

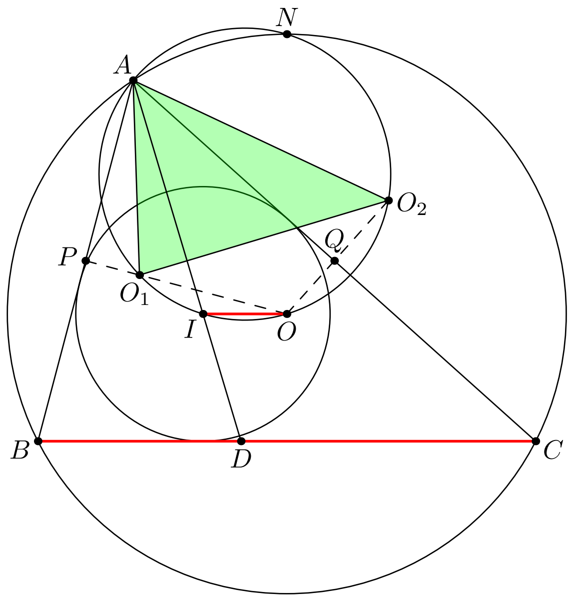
(The exact squares you get will depend on the order that you combine the terms.) Then, we find c and d such that $17547c - 6346d = 1$ using the Euclidean algorithm — one such pair is $(c, d) = (515, 1424)$. This allows us to get

$$(17547^2 + 6346^2)(515^2 + 1424^2) = (17547 \cdot 1424 + 6346 \cdot 515)^2 + 1,$$

which is 1 more than a square and a multiple of the product of the terms in the arithmetic sequence.

Problem 29. In $\triangle ABC$, the angle bisector of $\angle BAC$ intersects BC at D. Let I be the incenter of $\triangle ABC$, O_1 the circumcenter of $\triangle ABD$, O_2 the circumcenter of $\triangle ACD$, and O the circumcenter of $\triangle ABC$. It is given that A, I, O_1, and O_2 are concyclic. If the circumradius of $\triangle ABC$ is 10, the length of OI is 3, and the area of $\triangle ABC$ is 115, compute the area of $\triangle AO_1O_2$.

Proposed by Aditya Pahuja



Suppose without loss of generality that $AB < AC$, so that the configuration looks like the diagram above. Let P and Q be the midpoints of AB and AC respectively and let ω be the circumcircle of AO_1IO_2. The main fact that we will prove is

Claim 29.1 — Lines OI and BC are parallel.

Proof. To begin with, observe that $\angle AO_1O_2 = \frac{1}{2}\angle AO_1D = \frac{1}{2}\cdot 2\angle B = \angle B$, and analogously $\angle AO_2O_1 = \angle C$, so $\triangle ABC \sim \triangle AO_1O_2$. Thus, $\angle O_1AO_2 = \angle A$ and

$$\angle O_1OO_2 = \angle POQ = 180^\circ - \angle A.$$

This means that O also lies on ω. Moreover,

$$\begin{aligned}\angle O_1AO &= \angle OAB - \angle O_1AB \\ &= 90^\circ - \angle C - 90^\circ + \angle ADB \\ &= -\angle C + \frac{1}{2}\angle A + \angle C = \frac{1}{2}\angle A,\end{aligned}$$

so in fact AO is the bisector of $\angle O_1AO_2$.

Then, we have

$$\begin{aligned}\angle DIO = \angle AO_2O &= \angle AO_2O_1 + \angle O_1O_2O \\ &= \angle C + \frac{1}{2}\angle A = \angle IDB,\end{aligned}$$

which implies $IO \parallel BC$. □

Now, let N be the midpoint of major arc $\widehat{BAC}$ in the circumcircle of $\triangle ABC$. Since ON is the perpendicular bisector of line BC, we find that $OI \perp ON$, and because AN is the external bisector of $\angle A$, we have $AI \perp AN$. Thus, $AION$ is cyclic and N lies on ω.

Because $\triangle AO_1O_2 \sim \triangle ABC$, the ratio of their areas is the square of the ratio of their circumradii. The circumradius of $\triangle ABC$ is given to be 10, and the circumradius of $\triangle AO_1O_2$ is

$$\frac{IN}{2} = \frac{1}{2}\sqrt{OI^2 + ON^2} = \frac{1}{2}\sqrt{3^2 + 10^2} = \frac{\sqrt{109}}{2}.$$

Therefore,

$$[AO_1O_2] = \left(\frac{\sqrt{109}}{2 \cdot 10}\right)^2 \cdot [ABC] = \frac{115 \cdot 109}{400} = \boxed{\frac{2507}{80}}.$$

Remark 29.2. This configuration of points actually doesn't exist: the concyclic points, the circumradius, and the length of OI already uniquely determine the diagram, and result in $\triangle ABC$ having area $\frac{28119\sqrt{31719}}{43600}$.

Problem 30. For a positive integer k, let $\varphi(k)$ be the number of positive integers at most k and relatively prime to k. Find the sum of the first five odd numbers n for which $\varphi(n)$ has the same number of divisors as n.

Proposed by Aditya Pahuja

The answer is $1 + 3 + 15 + 255 + 65535 = \boxed{65809}$.

Let $\tau(n)$ be the number of divisors of n. We will exhaustively show that the five values above are the smallest working values of n by doing casework on the number of distinct prime factors of n. It will be helpful to reference the following table later on:

k	$\frac{k+1}{k}$
1	2
2	3/2
3	4/3
4	5/4

- If n has no prime factors, then $n = 1$, which works.
- If n has one prime factor, then it is a prime power p^a, having $a + 1$ factors. However, $\varphi(n) = (p - 1)p^{a-1}$ is a multiple of $2p^{a-1}$, meaning it has at least $2a$ factors, which forces $a = 1$ and thus n and $\varphi(n)$ must both be prime. This only happens at $n = 3$.
- If n has two prime factors, then say $n = p^a q^b$. We see that $2^2 p^{a-1} q^{b-1} \mid \varphi(n)$, so

  $$\tau(\varphi(n)) > 3ab,$$

  noting that equality cannot occur as this would require $p = q = 3$. Using the table above, we find that the inequality

  $$\tau(\varphi(n)) > 3ab > \tau(n) = (a + 1)(b + 1) \iff 3 > \frac{a + 1}{a} \cdot \frac{b + 1}{b}$$

  is true as long as (a, b) is not $(1, 1)$, which forces $(a, b) = (1, 1)$. Thus $n = pq$ has four factors, and so does $(p - 1)(q - 1)$. This means $(p - 1)(q - 1)$ is a product of two distinct primes or the cube of a prime. The former is impossible because $4 \mid (p - 1)(q - 1)$, and the latter implies $(p - 1)(q - 1) = 8$ so $n = 3 \cdot 5 = 15$.
- If $n = p^a q^b r^c$ has three prime factors $p < q < r$, then we similarly find $4abc$ as a lower bound for $\tau(\varphi(n))$. However, to optimize a bit, we will consider two subcases based on whether an odd prime divides $p - 1$.
  - Suppose $p - 1$ is divisible by an odd prime k. Since $k < p < q < r$, k cannot divide n, which means $\varphi(n)$ has a fourth odd prime factor and thus gives the stronger lower bound $\tau(\varphi(n)) > 8abc$. Since

    $$8 > \frac{a + 1}{a} \cdot \frac{b + 1}{b} \cdot \frac{c + 1}{c}$$

    is always true unless $(a, b, c) = (1, 1, 1)$, we have $n = pqr$.
  - Otherwise, $p - 1 = 2^k$ is a power of two.

    If $k = 1$, then $\tau(\varphi(n)) > 5abc$, which exceeds $\tau(n)$ unless (a, b, c) is some permutation of $(1, 1, 1)$, $(1, 1, 2)$, or $(1, 1, 3)$. In the latter two cases, say that t is the repeating prime, so that $n = pqrt$ or $n = pqrt^2$. For the first case, we see that $\tau(n) = 12$, so $\frac{\varphi(n)}{8} = t \cdot \frac{p-1}{2} \cdot \frac{q-1}{2} \cdot \frac{r-1}{2}$ is either 4 times a prime, the square of a prime, or 2^8, all of which are impossible. For the second case, $\frac{\varphi(n)}{8} = t^2 \cdot \frac{p-1}{2} \cdot \frac{q-1}{2} \cdot \frac{r-1}{2}$ is either a product of two primes, the cube of a prime, 16 times a prime, or 2^{12}, and again we can manually verify that none of these are possible. Therefore, $n = pqr$ is forced.

    If $k = 2$, then $\tau(\varphi(n)) > 6abc$, forcing $(a, b, c) = (1, 1, 1)$ again.

    Otherwise, if $k = 3$, $p = 2^3 + 1$ isn't prime, and $k = 4$ gives

    $$\tau(\varphi(n)) > 8abc \geq (a + 1)(b + 1)(c + 1),$$

    so no other k give a working p.

Each of these cases ends up with $n = pqr$, so $\tau(n) = 8$. Then $\varphi(n) = (p-1)(q-1)(r-1)$ is either 8 times a prime or a power of 7. The former is impossible, and the latter gives

$$(p-1)(q-1)(r-1) = 128,$$

implying $n = 3 \cdot 5 \cdot 17 = 255$.

- If $n = p^a q^b r^c s^d$ has four prime factors, then we optimize as above, but twice — that is, we have four cases based on whether p and q contribute new prime factors. We will glaze over the casework because it is quite frankly really annoying to write out, but you should find $n = 65535$ as the next working n.
- If n has five prime factors, then each prime factor has an exponent of 1 with possibly the exception of 3; that is, the prime factorization is either $3pqrs$ or 3^2pqrs (if its smallest prime factor is 5, then $n \geq 85085$, which is too big).

  In the first case, n has 32 factors. The idea is then to casework on p, supposing that $p < q < r < s$. We can check $p = 11$ is too large, so $p = 5$ or $p = 7$. At this point, it's straightforward (but kind of annoying) to check that there are no working n that are smaller than 65536.

  In the second case, we only have sufficiently small n, namely $45045 = 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ and $58905 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17$. Neither works.
- If n has at least six prime factors, then $n \geq 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 255255$, so this case contributes nothing.

This concludes our bash, and gives us the five values of n above.

Remark 30.1 (A lesson in testsolving)**.** This problem has a very ugly solution that is unsuitable for any reasonable contest. The originally intended solution was to prove the claim $\tau(\varphi(n)) \geq \tau(n)$ for all odd n, which is true for $n \leq 2 \cdot 10^4$ according to Python. However, it does not extend to all odd n — had Aditya checked up to $n = 7 \cdot 10^4$ (which is not too large, since the algorithm for checking from 1 to n is only $O(n\sqrt{n})$), he would've found $n = 69615$, which is a counterexample to the inequality. In fact, there are 11 counterexamples to the inequality for $n < 10^6$.

Aditya also conjectured that the complete solution set to the original problem was $n = 2^{2^k} - 1$ for $0 \leq k \leq 5$, to which the first counterexample is $n = 77805$.