NYCMT 2023-2024 Homework #4 Solutions

NYCMT

December 2023

Problem 1. Find the last two digits of $1^{2023} - 2^{2023} + 3^{2023} - 4^{2023} + \cdots + 9^{2023} - 10^{2023}$.

Answer. $\boxed{75}$

Solution. Call the given value N. In order to find the last two digits, we need to find $N \pmod{100}$, or, equivalently, $N \pmod 4$ and $N \pmod{25}$.

Note that all even terms are raised to the 2023rd power, and are clearly divisible by 4. This means that we can ignore them and write

$$N \equiv 1^{2023} + 3^{2023} + 5^{2023} + 7^{2023} + 9^{2023} \equiv 1 + 3 + 1 + 3 + 1 \equiv 1 \pmod 4.$$

Similarly, we can ignore multiples of 5 when computing $N \pmod{25}$, so

$$N \equiv 1^{2023} - 2^{2023} + 3^{2023} - 4^{2023} - 6^{2023} + 7^{2023} - 8^{2023} + 9^{2023} \pmod{25}.$$

By Euler's Theorem, $a^{\varphi(25)} \equiv 1 \pmod{25}$ for all a not divisible by 5. $\varphi(25) = 20$, meaning $a^{20} \equiv a^{2020} \equiv 1 \pmod{25}$. We can now rewrite

$$N \equiv 1^3 - 2^3 + 3^3 - 4^3 - 6^3 + 7^3 - 8^3 + 9^3 = 300 \equiv 0 \pmod{25}.$$

By the Chinese Remainder Theorem, $N \equiv 25 \pmod{100}$. However, 25 is not the answer! We have to check the sign of N, and, since the last term is -10^{2023}, N is negative. When N is negative, we use the negative residue, $-75 \pmod{100}$, meaning its last two digits are $\boxed{75}$, as desired.

Note. Instead of computing each term mod 100 directly, we may pair up k^{2023} and $(10-k)^{2023}$ together. From binomial expansion, the only relevant term is $\binom{2023}{1}(10^1)(k^{2022})$, which does not require Euler's Theorem to compute mod 100. $\square$

Problem 2. Mr. Sterr and Mr. Kats are both standing on space 1 of a board game with spaces numbered 1, 2, 3 . . . 35 in that order. The board loops around, so space 1 is directly after space 35. Both players roll a fair 35-sided die with faces numbered 1 to 35. Then, they each square the number rolled and move forward that many spaces. What is the probability that Mr. Kats is on a higher-numbered space than Mr. Sterr?

Answer. $\boxed{\frac{554}{1225}}$

Solution. Let S be the number that Mr. Sterr rolls, and K be the number that Mr. Kats rolls. Subtract 1 from each board space, so that they are now numbered 0 through 34. When a player moves X spaces, since every 35 spaces loops back to the same spot, they will land on space $X \pmod{35}$.

Then, we want to compute the probability p that $K^2 > S^2 \pmod{35}$. Since this is symmetric to $S^2 > K^2 \pmod{35}$, we can say that $1 = 2p + s$, where s is the probability that $K^2 \equiv S^2 \pmod{35}$. We will compute s.

By the Chinese Remainder Theorem, $a \equiv b \pmod{35}$ is equivalent to $a \equiv b \pmod 5$ and $a \equiv b \pmod 7$. We can list out the quadratic residues for each prime in a table:

n	$n^2 \pmod 5$	$n^2 \pmod 7$
0	0	0
1	1	1
2	4	4
3	4	2
4	1	2
5		4
6		1

We see that the quadratic residues residues mod 5 are one 0, two 1s, and two 4s. This means that the probability that $K^2 \equiv S^2 \pmod 5$ is

$$\frac{1^2 + 2^2 + 2^2}{5^2} = \frac{9}{25}.$$

Similarly, the quadratic residues mod 7 are one 0, two 1s, two 2s, and two 4s. Then, the probability that $K^2 \equiv S^2 \pmod 7$ is

$$\frac{1^2 + 2^2 + 2^2 + 2^2}{7^2} = \frac{13}{49}.$$

The probability that $K^2 \equiv S^2 \pmod{35}$ is the product of these two:

$$s = \frac{9}{25} \cdot \frac{13}{49} = \frac{117}{1225}.$$

Finally, we can solve for p:

$$p = \frac{1 - s}{2} = \frac{1225 - 117}{2 \cdot 1225} = \boxed{\frac{554}{1225}}$$

as desired. □

Problem 3. Let the sequence a_n be defined recursively as $a_1 = 1$ and $a_{n+1} = (n+1) \cdot 1000^n + a_n$ for all $n \geq 1$. Find the sum of the digits of $999^2 \cdot a_{2023}$.

Answer. $\boxed{27}$

Solution. Let $x = 1000$. Then,

$$\begin{aligned}
a_n &= \sum_{k=1}^{n} kx^{k-1} = 1 + 2x + 3x^2 + \cdots + (n-1)x^{n-2} + nx^{n-1} \\
&= x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1 \\
&+ x^{n-1} + x^{n-2} + \cdots + x^2 + x \\
&+ x^{n-1} + x^{n-2} + \cdots + x^2 \\
&\vdots \\
&+ x^{n-1} + x^{n-2} \\
&+ x^{n-1} \\
&= \frac{x^n - 1}{x-1} + \frac{x^n - x}{x-1} + \frac{x^n - x^2}{x-1} + \cdots + \frac{x^n - x^{n-2}}{x-1} + \frac{x^n - x^{n-1}}{x-1} \\
&= \frac{nx^n - (x^{n-1} + x^{n-2} + \cdots + x^2 + x + 1)}{x-1} \\
&= \frac{nx^n - \frac{x^n - 1}{x-1}}{x-1} \\
&= \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2} \\
&= \frac{n \cdot 1000^{n+1} - (n+1) \cdot 1000^n + 1}{999^2}
\end{aligned}$$

is the closed form for a_n. Then,

$$\begin{aligned}
999^2 \cdot a_{2023} &= 2023 \cdot 1000^{2024} - 2024 \cdot 1000^{2023} + 1 \\
&= (2023000 - 2024) \cdot 1000^{2023} + 1 \\
&= 2020976000 \ldots 001,
\end{aligned}$$

which has a digit sum of $2 + 2 + 9 + 7 + 6 + 1 = \boxed{27}$ as desired.

Note. The closed form can be more quickly computed using calculus... though calculus is not required for this problem (nor any problems on NYCMT homework).

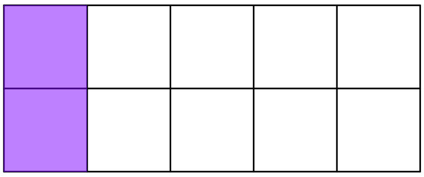
Note. There exist solutions using induction. □

Problem 4. How many ways are there to tile a 2×5 board with any number of 1×2 dominoes and 1×4 tetrominoes?

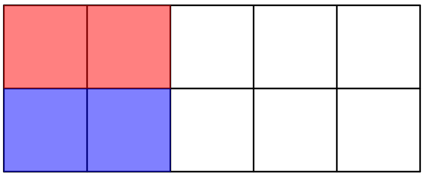
Answer. $\boxed{14}$

Solution. We can casework on how many 1×4 tetrominoes are used.

Case 1: No tetrominoes are used. Then, define a_n to be the number of ways to tile a $2 \times n$ board (two rows of n cells) with 1×2 dominoes, and it is clear that $a_1 = 1$ and $a_2 = 2$. There are two ways to place dominoes covering the top-left corner (see diagrams): vertically, which reduces the $2 \times n$ board to a $2 \times n - 1$ board, or horizontally, forcing another domino below it, which reduces the $2 \times n$ board to a $2 \times n - 2$ board.



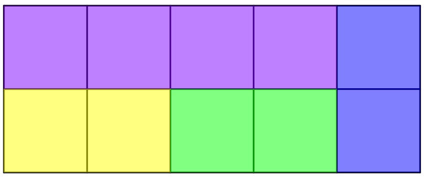
Vertical Domino



Horizontal Dominoes

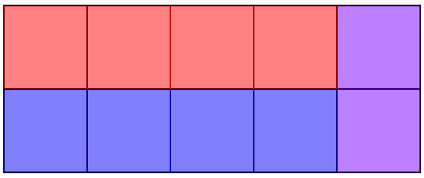
This gives the recursive relation $a_n = a_{n-1} + a_{n-2}$. We can compute $a_3 = 3$, $a_4 = 5$, and $a_5 = 8$. This case gives us 8 tilings.

Case 2: One tetromino is used. WLOG let it cover the top-left corner. Then, there is exactly one way to tile the other six cells with dominoes, as shown in the diagram below.

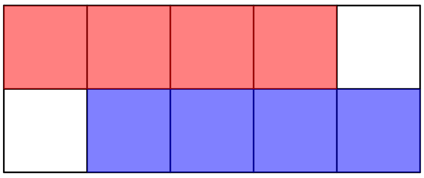


This case gives us 4 tilings; one for each orientation of the tetromino.

Case 3: Two tetrominoes are used. It is clear that the two tetrominoes must be vertically aligned in order for the last two cells to be adjacent.



Valid Tiling



Invalid Tiling

This case gives us 2 tilings; one for each orientation of the domino.

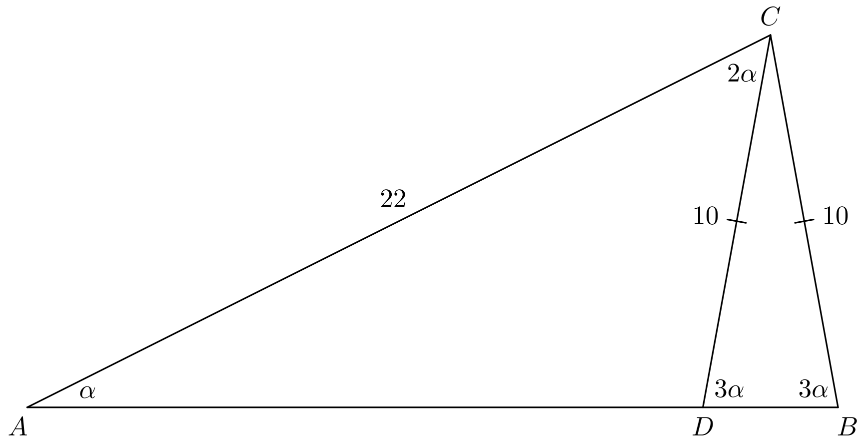
Since it is impossible to use three or more tetrominoes, the answer is $8 + 4 + 2 = \boxed{14}$.

Note. It is possible to construct a recursion that takes into account both dominoes and tetrominoes. However, it is a strong recursion; that is, a_{n+1} can be written as a linear combination of a_i where $0 \le i \le n$. However, for the small case of computing a_5, I accepted all recursions that considered terms down to a_{n-4}. $\square$

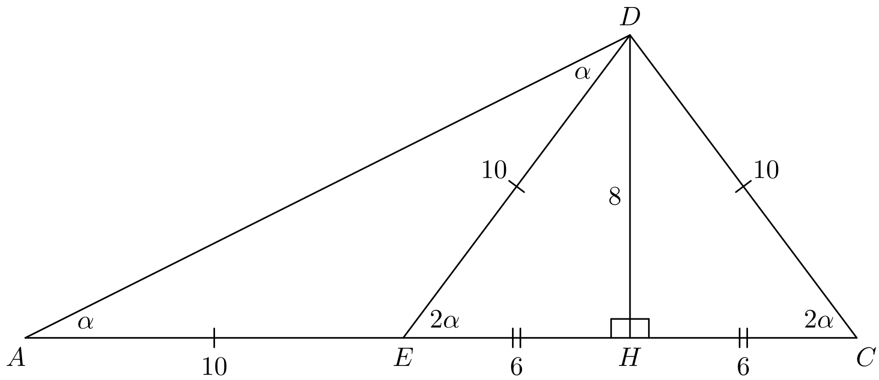
Problem 5. In $\triangle ABC$, $m\angle B = 3m\angle A$, $AC = 22$, and $BC = 10$. Point D is drawn on $\overline{AB}$ such that $m\angle ACD = 2m\angle A$. Find the area of $\triangle ACD$.

Answer. $\boxed{88}$

Solution. Let $m\angle A = \alpha$. Then, $m\angle CDB = m\angle A + m\angle ACD = \alpha + 2\alpha = 3\alpha$ because $\angle CDB$ is exterior to $\triangle ACD$. This means $\triangle BCD$ is isosceles, and $CD = BD = 10$.



Now, we focus on $\triangle ACD$. Let H be the foot of the altitude from D to $\overline{AC}$, and let E be the reflection of D over H. We aim to find DH.



Then, $\triangle DHC \cong \triangle DHE$ and $m\angle DEH = m\angle DCH = 2\alpha$. Since $\angle DEH$ is exterior to $\triangle ADE$, $m\angle ADE = m\angle DEH - m\angle A = 2\alpha - \alpha = \alpha$. This means $\triangle ADE$ and $\triangle CDE$ are both isosceles, so $CD = DE = EA = 10$. And, since $EH = HC$, we have $AE + EH + HC = 22 = 10 + 2EH \implies EH = HC = 6$, so $DH = 8$ by the Pythagorean Theorem. Finally, $[ADC] = \frac{1}{2} \cdot AC \cdot DH = \frac{1}{2} \cdot 22 \cdot 8 = \boxed{88}$ as desired.

Note. There exist solutions using similar triangles, found by drawing the angle bisector of $\angle ACD$.

Note. There exist solutions using the double angle ($\sin 2\alpha = 2 \sin\alpha \cos\alpha$) or triple angle ($\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$) formulas. However, it should be appreciated that nothing beyond 9th grade geometry is necessary to solve this problem! $\square$

Problem 6.

Reduce the following numbers:

1. The answer to Problem 1, expressed as a two-digit number, multiplied by 2.
2. The numerator of the answer to Problem 2 (provided the answer is in lowest terms).
3. The answer to Problem 3.
4. The answer to Problem 4.
5. The answer to Problem 5, plus 1.
6. 20232023.

Answer. $\boxed{\text{THANKS}}$

Solution. The desired quantities are as follows:

1. 150
2. 554
3. 27
4. 14
5. 89
6. 20232023

As per the global hint that was sent out at noon on November 21st, these numbers should be reduced mod 26 to get

1. 20
2. 8
3. 1
4. 14
5. 11
6. 19

The number 26 hints that these numbers should be used as indices into the English alphabet, with 1 corresponding to A, 2 to B, and so on. Our final answer is $\boxed{\text{THANKS}}$, a fitting answer for the time of year.

Note. I received some complaints about releasing a global hint for the puzzle, because most of you procrastinate the homework anyway. Maybe next time, I'll make the puzzle harder, offer a prize, and give no hints. Then we'll see who procrastinates.

Note. Seasoned puzzle hunters might dislike my method of extraction, as reducing numbers mod 26 gives integers in the range $[0, 25]$, not $[1, 26]$. Still, the answer does not include the letter Z, so I thought it appropriate and more beginner-friendly to directly correspond the residues to their alphabetical counterparts. $\square$