NYCMT 2023-2024 Playoffs Solutions

NYCMT

October 2023

Problem 1. If $a^3 + a - 1 = 0$, find $a^5 - a^2 - a + 6$.

Answer. $\boxed{5}$

Solution. In order to avoid solving the cubic, we can rewrite the first equation as

$$a^3 = 1 - a$$

and substitute wherever appropriate to reduce the degree of the expression we want to find. Since $a^5 = a^3 \cdot a^2$, we see that

$$a^5 - a^2 - a + 6 = (1 - a)a^2 - a^2 - a + 6 = -a^3 - a + 6.$$

Substituting again, we have

$$-a^3 - a + 6 = -(1 - a) - a + 6 = -1 + 6 = \boxed{5}$$

as desired. □

Problem 2. How many ordered pairs of integers are there such that the absolute difference between their product and their sum is 35?

Answer. $\boxed{26}$

Solution. We want the number of integer solutions to $|xy - x - y| = 35$, which can be split into two cases.

Case 1: $xy - x - y = 35$. Adding 1 to both sides and factoring using Simon's Favorite Factoring Trick produces

$$xy - x - y + 1 = 36 = (x-1)(y-1).$$

Since $y - 1$ is determined by $x - 1$, and $x - 1$ can be any integer that divides 36, the number of solutions (x, y) to this equation is the number of factors of 36, both positive AND negative. We know $36 = 2^2 \cdot 3^2$ has 9 factors, so this case has $2 \cdot 9 = 18$ solutions.

Case 2: $xy - x - y = -35$. We do the same thing:

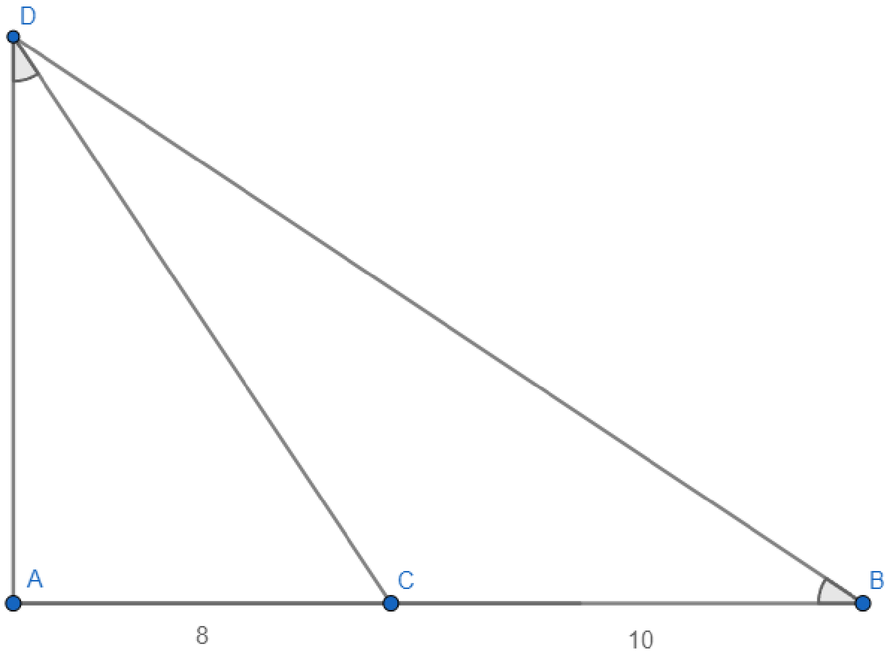
$$xy - x - y + 1 = -34 = (x-1)(y-1).$$

Similarly, we want the number of factors of -34, both positive AND negative. Since $34 = 2 \cdot 17$ has four factors, this case has $2 \cdot 4 = 8$ solutions.
These cases give a total of $18 + 8 = \boxed{26}$ solutions, as desired. □

Problem 3. Point C lies on segment $\overline{AB}$ such that $AC = 8$ and $CB = 10$. Point D lies in the plane such that $\angle ADC \cong \angle DBC$. Find the maximum possible area of $\triangle DAB$.

Answer. $\boxed{108}$

Solution. Since $\angle CAD \cong \angle DAB$, $\triangle CAD \sim \triangle DAB$ by AA similarity.



Then, $\frac{CA}{AD} = \frac{AD}{BA} = \frac{8}{AD} = \frac{AD}{18}$, so $AD = 12$. Since this is the only necessary condition, D can be any point on the circle centered at A with radius 12. We know that $[DAB] = \frac{1}{2} \cdot AB \cdot h$, where h is the length of the altitude from D to $\overline{AB}$. h is maximized when $\overline{AD} \perp \overline{AB}$, or $h = AD = 12$. Then, the area is $\frac{1}{2} \cdot 18 \cdot 12 = \boxed{108}$, as desired. $\square$

Problem 4. How many ways are there to arrange the numbers $1, 2, 3, \ldots, 9$ in a row such that, for any two integers a and b, if $a \mid b$, then a comes before b?

Answer. $\boxed{1960}$

Solution. It is clear that 1 must be the first number, and that 5 and 7 may be inserted anywhere afterwards, as they do not depend on nor affect any other numbers. So, we will first focus on arranging 2, 3, 4, 6, 8, and 9.

Let's start with the sequence $(2, 4, 8)$ because the powers of 2 must occur in that order. Then, placing 3 determines where 6 and 9 may go. We will casework on where 3 appears relative to the powers of 2.

Case 1: If 3 appears before 2, resulting in $(3, 2, 4, 8)$, then 6 has three possible spots; right after the 2, 4, or 8. Once 6 is placed, 9 has five possible spots; anywhere that is not before 3. This gives $3 \cdot 5 = 15$ sequences.

Case 2: If 3 appears between 2 and 4, resulting in $(2, 3, 4, 8)$, 6 still has three possible spots, but now, once 6 is placed, 9 has four spots. This gives $3 \cdot 4 = 12$ sequences.

Case 3: If 3 appears between 4 and 8, 6 has two possible spots, and once it is placed, 9 has three. This gives $2 \cdot 3 = 6$ sequences.

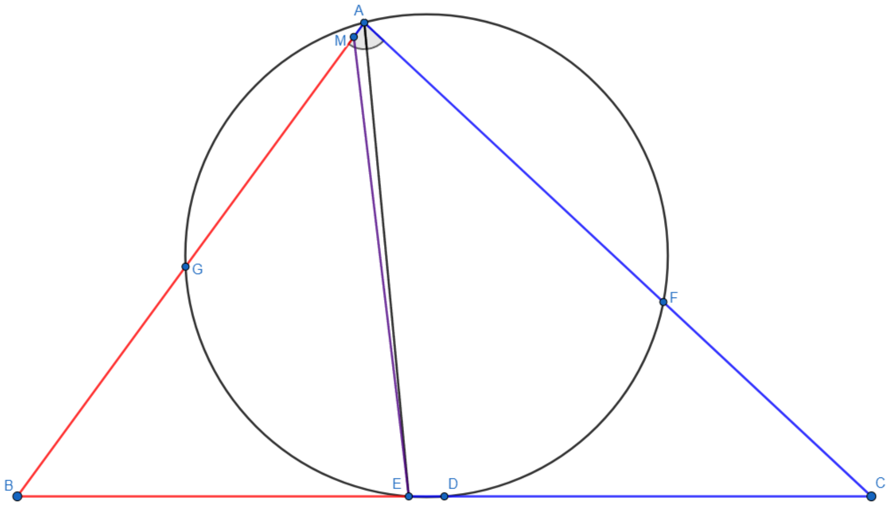
Case 4: If 3 appears after 8, there is only one spot to place 6; at the end, and that gives two spots to place 9. This gives $1 \cdot 2 = 2$ sequences.

The total number of ways to arrange 2, 3, 4, 6, 8, and 9 is $15 + 12 + 6 + 2 = 35$. Finally, we multiply by 7 to place 5, and then multiply by 8 to place 7. Our final answer is $35 \cdot 7 \cdot 8 = \boxed{1960}$ as desired. $\square$

Problem 5. Suppose $\triangle ABC$ has side lengths $AB = 11$, $BC = 16$. and $CA = 13$. Let D be the midpoint of $\overline{BC}$ and E be the point where the angle bisector of A intersects $\overline{BC}$. Let the circumcircle of $\triangle ADE$ meet $\overline{AC}$ and $\overline{AB}$ again at F and G respectively. Suppose $M \neq B$ is a point on $\overline{AB}$ such that $MG = GB$. Find the ratio $\frac{[BME]}{[CAME]}$.

Answer. $\boxed{\frac{4}{5}}$

Solution. We will find the ratio of the areas by computing most of the relevant side lengths.



By the Angle Bisector Theorem,

$$\frac{EB}{AB} = \frac{EC}{AC} = \frac{EB}{11} = \frac{EC}{13}.$$

And since $EB + EC = BC = 16$, solving the system gives $EB = \frac{22}{3}$ and $EC = \frac{26}{3}$. We also know that D is the midpoint of $\overline{BC}$, meaning $CD = 8$ and $DE = \frac{26}{3} - 8 = \frac{2}{3}$.

Now, we apply Power of a Point on point B with respect to the circumcircle. That is,

$$BG \cdot BA = BE \cdot BD = 11 \cdot BG = \frac{22}{3} \cdot 8$$

which gives $BG = \frac{16}{3}$. Finally, since G is the midpoint of BM, $BM = 2BG = \frac{32}{3}$. Now,

$$\frac{[BME]}{[ABC]} = \frac{BE}{BC} \cdot \frac{BM}{BA} = \frac{\frac{22}{3}}{16} \cdot \frac{\frac{32}{3}}{11} = \frac{4}{9}$$

and $\frac{[CAME]}{[ABC]} = 1 - \frac{4}{9} = \frac{5}{9}$, so our answer is

$$\frac{[BME]}{[CAME]} = \frac{\frac{[BME]}{[ABC]}}{\frac{[CAME]}{[ABC]}} = \frac{\frac{4}{9}}{\frac{5}{9}} = \boxed{\frac{4}{5}}$$

as desired. □

Problem 6. Let A be the number of functions $f : \{1,2,3,4,5,6,7,8,9\} \longrightarrow \{1,2,3,4,5,6,7,8,9\}$ such that n and $f(n)$ are different parity and $f(f(n)) \neq n$ for all $1 \leq n \leq 9$. Find the remainder when A is divided by 1000.

Answer. $\boxed{280}$

Solution. We will use Principle Inclusion-Exclusion to solve this problem. First, only consider the condition n and $f(n)$ have different parities. Then, there are 4 even numbers for each of the 5 odd numbers to go to, and 5 odd numbers for each of the 4 even numbers to go to, so there are $4^5 \cdot 5^4 = 640000$ total functions f satisfying this condition.

Now, we want to subtract the overcount, when $f(f(n)) = n$ for at least 1 value of n. However, these values of n must come in pairs. If $f(a) = b$ and $f(b) = a$, then $n = a$ and $n = b$ both satisfy the condition. Furthermore, a and b must be of different parities. Thus, by PIE, the overcount is equal to

$$\binom{5}{1}\binom{4}{1} \cdot k_1 - \binom{5}{2}\binom{4}{2} \cdot 2! \cdot k_2 + \binom{5}{3}\binom{4}{3} \cdot 3! \cdot k_3 - \binom{5}{4}\binom{4}{4} \cdot 4! \cdot k_4$$

where k_x is the number of functions f given x pairs of (a, b) that satisfy $f(a) = b$ and $f(b) = a$, and their respective coefficients represent the number of ways to choose x pairs of integers that are of different parities. Then, $k_x = 4^{(5-x)} \cdot 5^{(4-x)}$ because there are $5 - x$ odd numbers that are not yet fixed, and $4 - x$ even numbers that are not yet fixed. Evaluating, the overcount is equal to

$$5 \cdot 4 \cdot 4^4 \cdot 5^3 - 10 \cdot 6 \cdot 2 \cdot 4^3 \cdot 5^2 + 10 \cdot 4 \cdot 6 \cdot 4^2 \cdot 5^1 - 5 \cdot 1 \cdot 24 \cdot 4^1 \cdot 5^0$$

$$= 640000 - 192000 + 19200 - 480$$

$$= 466720.$$

This means that $A = 640000 - 466720 = 173280$, which has a remainder of $\boxed{280}$ when divided by 1000, as desired.

$\square$

Problem 7. Find all real x that satisfy the equation $2x\sqrt{1-x^2}+2x^2-\sqrt{2}x-1=0$.

Answer. $\boxed{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{6}+\sqrt{2}}{4}}$

Solution 1. Since $\sqrt{1-x^2}$ being defined implies $x \in [-1,1]$, it makes sense to do a trigonometric substitution. Let $x = \cos(\theta)$. (A similar solution exists for $x = \sin(\theta)$.) Then, $\sqrt{1-x^2} = \sqrt{1-\cos^2(\theta)} = \sqrt{\sin^2(\theta)} = |\sin(\theta)|$.

Note. Since x is defined in terms of cosine, we can, WLOG, restrict $\theta \in [0,\pi]$, as that interval achieves the full range of cosine. This implies $\sin(\theta) \geq 0$, which eliminates the absolute value. This also means that we have to make sure our solutions for θ are in that interval to eliminate extraneous solutions.

With this, the equation becomes

$$2\cos(\theta)\sin(\theta) + 2\cos^2(\theta) - \sqrt{2}\cos(\theta) - 1 = 0.$$

We can use the double angle formulas and move the rest to the right side:

$$\sin(2\theta) + \cos(2\theta) = \sqrt{2}\cos(\theta).$$

Dividing by $\sqrt{2}$ results in a cosine subtraction:

$$\frac{\sqrt{2}}{2}\sin(2\theta) + \frac{\sqrt{2}}{2}\cos(2\theta) = \cos(\theta)$$
$$\sin\left(\frac{\pi}{4}\right)\sin(2\theta) + \cos\left(\frac{\pi}{4}\right)\cos(2\theta) = \cos(\theta)$$
$$\cos\left(2\theta - \frac{\pi}{4}\right) = \cos(\theta).$$

We now have an equation of the form $\cos(a) = \cos(b)$, which is true when $a = b$ or $a = -b$ up to adding multiples of 2π. That is, $a - b = 2\pi k$ or $a + b = 2\pi k$ for some integer k. Applying this gives

$$\theta - \frac{\pi}{4} = 2\pi k \text{ OR } 3\theta - \frac{\pi}{4} = 2\pi k$$

$$\theta = \frac{\pi}{4} + 2\pi k \text{ OR } \theta = \frac{\pi}{12} + \frac{2\pi}{3}k$$
$$\theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{17\pi}{12}.$$

The last solution for θ is not in $[0,\pi]$, so it is extraneous. The other three produce $x = \cos\left(\frac{\pi}{12}\right), \cos\left(\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) = \boxed{\frac{\sqrt{6}+\sqrt{2}}{4}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}}$ as desired. $\square$

Solution 2. Isolating the radical on the left and squaring both sides yields

$$\left(2x\sqrt{1-x^2}\right)^2 = \left(-2x^2+\sqrt{2}x+1\right)^2$$

$$\begin{aligned} 4x^2-4x^4 &= 4x^4-4\sqrt{2}x^3-2x^2+2\sqrt{2}x+1 \\ 0 &= 8x^4-4\sqrt{2}x^3-6x^2+2\sqrt{2}x+1. \end{aligned}$$

Note. Squaring the equation loses information about the restrictions on x (specifically, x and $-2x^2+\sqrt{2}x+1$ must be the same sign), so we will need to plug our solutions back in to verify that they are not extraneous.

Since the irrational coefficients occur on odd-degree terms, we can let $x = k\sqrt{2}$, and substituting results in a polynomial in k with integer coefficients:

$$8\left(k\sqrt{2}\right)^4-4\sqrt{2}\left(k\sqrt{2}\right)^3-6\left(k\sqrt{2}\right)^2+2\sqrt{2}\left(k\sqrt{2}\right)+1=0$$

$$32k^4-16k^3-12k^2+4k+1=0.$$

The Rational Root Theorem produces $k=\pm\frac{1}{2}$ as solutions, so we factor them out:

$$(2k+1)(2k-1)(8k^2-4k-1)=0.$$

The quadratic then has zeroes $k=\frac{-(-4)\pm\sqrt{(-4)^2-4(8)(-1)}}{2(8)}=\frac{1\pm\sqrt{3}}{4}$. So, $x=k\sqrt{2}$ has solutions $\pm\frac{\sqrt{2}}{2}, \frac{\sqrt{2}\pm\sqrt{6}}{4}$. Plugging these back in, it turns out that $x=\frac{\sqrt{2}-\sqrt{6}}{4}$ is extraneous, while the other three work. This produces $\boxed{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}+\sqrt{6}}{4}}$ as our answer, as desired. □

Problem 8. Compute the sum of all prime numbers p such that $37p^2 - 47p + 4$ is the square of an integer.

Answer. $\boxed{97}$

Solution 1. We want to find all primes p such that $37p^2 - 47p + 4 = n^2$ for some integer n. Subtracting 4 from both sides gives

$$37p^2 - 47p = n^2 - 4$$

$$p(37p - 47) = (n - 2)(n + 2).$$

Since p divides the left side, it must also divide the right side, meaning $p \mid n - 2$ or $p \mid n + 2$. Equivalently, $n \equiv 2 \pmod{p}$ or $n \equiv -2 \pmod{p}$.

Note that $37p^2 - 47p - 4 \approx 36p^2 = (6p)^2$. So, we should test $n = 6p \pm 2$.

$$37p^2 - 47p + 4 = (6p - 2)^2 = 36p^2 - 24p + 4$$

$$p^2 - 23p = 0$$

$$p = 23.$$

$$37p^2 - 47p + 4 = (6p + 2)^2 = 36p^2 + 24p + 4$$

$$p^2 - 71p = 0$$

$$p = 71.$$

Both 23 and 71 are prime, so they are valid solutions.

We now aim to show that

$$(5p + 2)^2 < 37p^2 - 47p + 4 < (7p - 2)^2$$

for sufficiently large p, as $n = 5p + 2$ and $n = 7p - 2$ are the next smallest and next largest possibilities respectively.

The left inequality is true when $p > 6 \implies 12p^2 > 72p \implies 37p^2 - 47p > 25p^2 + 25p \implies 37p^2 - 47p + 4 > 25p^2 + 25p + 4 > 25p^2 + 20p + 4 = (5p + 2)^2$.

The right inequality is always true, since p is positive: $37p^2 - 47p + 4 < 49p^2 - 28p + 4$ because $37p^2 < 49p^2$ and $-47p < -28p$.

This means we need to check $p = 2$, $p = 3$, and $p = 5$ manually. $p = 3$ gives $n = 14$, while the other two don't produce integer n. This gives us three solutions in total with a sum of $3 + 23 + 71 = \boxed{97}$, as desired.

□

Solution 2. We get $n \equiv 2 \pmod{p}$ or $n \equiv -2 \pmod{p}$ as from Solution 1. Consider the first case and set $n = kp + 2$ for some non-negative integer k. Then, plugging in gives

$$p(37p - 47) = (kp)(kp + 4)$$

$$37p - 47 = k^2p + 4k$$

$$p = \frac{4k + 47}{37 - k^2}.$$

We can check all values of k that make the above expression positive, which would be $0 \leq k \leq 6$. $k = 4 \implies p = 3$ and $k = 6 \implies p = 71$ both work, as 3 and 71 are prime, while the others do not produce integer p.

Now, we consider the second case, setting $n = kp-2$ for some positive integer k. Plugging in gives

$$p(37p - 47) = (kp - 4)(kp)$$

$$37p - 47 = k^2p - 4k$$

$$p = \frac{4k - 47}{k^2 - 37}.$$

Now, the values of k that make the above expression positive are $1 \leq k \leq 6$ OR $k \geq 12$; however, $k \geq 12$ clearly has $k^2 - 37 > 4k - 47$, meaning p is never an integer in that case. We check the other six possibilities manually, giving $k = 6 \implies p = 23$ as the only solution.

So, the sum of our solutions is $3 + 71 + 23 = \boxed{97}$, as desired. $\square$

Problem 9. Find the number of sequences of integers $(a_0, a_1, \ldots, a_{50})$ with $0 \leq a_n \leq 5$ such that:

$$a_0 + a_1 \cdot 4 + a_2 \cdot 4^2 + \ldots + a_{50} \cdot 4^{50} = 2023 \cdot 4^{45}$$

Answer. $\boxed{138}$

Solution. Let S be the sum of the first 45 terms on the left, and T be the sum of the remaining 6. Then, we have $S + T = 2023 \cdot 4^{45}$. Since the right side is divisible by 4^{45}, the left side must also be. And because T consists only of terms $a \cdot 4^k$ where $k \geq 45$, that means that $4^{45} \mid S$.

Now, let a_z be the first non-zero term of the sequence. That is, let z be the unique integer such that $a_0 = a_1 = \ldots a_{z-1} = 0$, and $a_z \neq 0$.

If $z > 44$, $a_0 = a_1 = \cdots = a_{44} = 0 = S$, which gives one case, and T must be $2023 \cdot 4^{45}$.

If $z \leq 44$, then $a_z = 4$; otherwise, $4^{z+1} \nmid S$. Furthermore, $a_{z+1} = a_{z+2} = \ldots a_{44} = 3$, because if any term $a_i \neq 3$ for $i \leq 44$, $4^{i+1} \nmid S$. This produces 45 cases, as $0 \leq z \leq 44$. This means that

$$S = 4 \cdot 4^z + \sum_{i=z+1}^{44} 3 \cdot 4^i = 4^{45} - 4^{z+1} + 4^{z+1} = 4^{45}.$$

For these 45 cases, T must be $2022 \cdot 4^{45}$.

Now, we manually check how many ways we can construct T by looking at $T \pmod{4^i}$ for $46 \leq i \leq 50$. Take the case where $T = 2022 \cdot 4^{45}$. Because $T \equiv 2 \cdot 4^{45} \pmod{4^{46}}$, a_{45} must be 2, as all other terms are $0 \pmod{4^{46}}$. Then, $T - 2 \cdot 4^{45} = 2020 \cdot 4^{45} = 505 \cdot 4^{46}$, which is $1 \cdot 4^{46} \pmod{4^{47}}$. As a result, a_{46} must be either 1 or 5. Continuing, we get that there are 3 possible sequences $(a_{45}, a_{46}, a_{47}, a_{48}, a_{49}, a_{50})$, namely $(2, 1, 2, 3, 3, 1)$, $(2, 5, 1, 3, 3, 1)$, and $(2, 5, 5, 2, 3, 1)$. A very similar process for $T = 2023 \cdot 4^{45}$ yields the same three sequences, with $a_{45} = 3$ instead of 2.

This means that there are $3 \cdot 45 + 3 \cdot 1 = \boxed{138}$ total sequences, as desired.

Note. The process of checking S and $T \pmod{4^i}$ is analogous to writing $2023 \cdot 4^{45}$ in base 4, except $4 = 10_4$ and $5 = 11_4$ are also allowed "digits". $\square$

Problem 10. Let $A = \sum_{n=1}^{99} \sqrt{10 + \sqrt{n}}$ and $B = \sum_{n=1}^{99} \sqrt{10 - \sqrt{n}}$. Find $\frac{B}{A}$.

Answer. $\boxed{\sqrt{2} - 1}$

Solution. Consider the sum

$$A + B = \sum_{n=1}^{99} \sqrt{10 + \sqrt{n}} + \sqrt{10 - \sqrt{n}}.$$

We can rewrite this as

$$\sum_{n=1}^{99} \sqrt{\left(\sqrt{10 + \sqrt{n}} + \sqrt{10 - \sqrt{n}}\right)^2} = \sum_{n=1}^{99} \sqrt{20 + 2\sqrt{100 - n}}$$

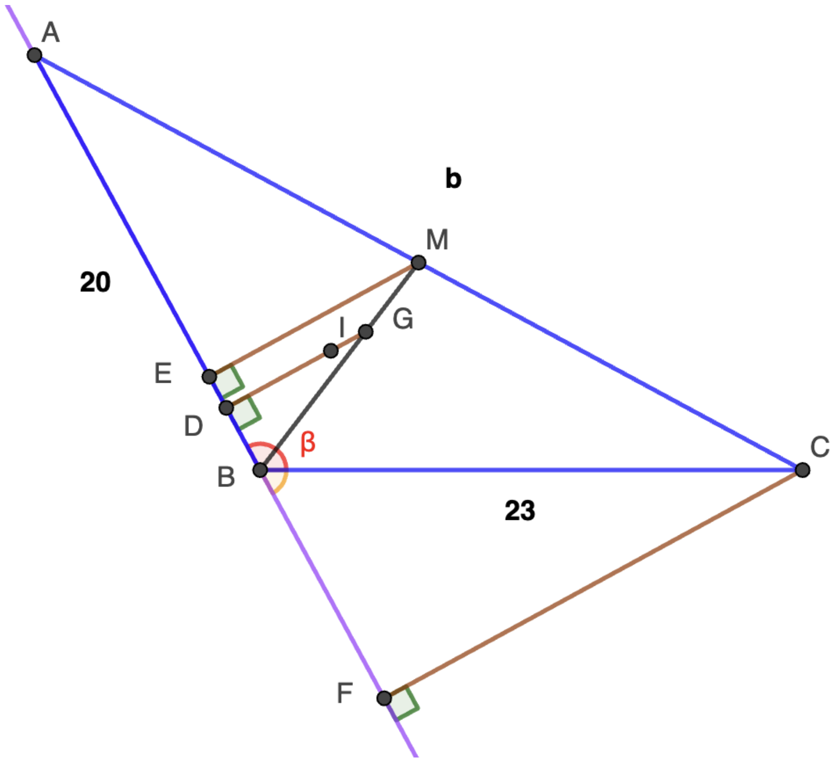
$$= \sqrt{2} \cdot \sum_{n=1}^{99} \sqrt{10 - \sqrt{100 - k}} = \sqrt{2} \cdot \sum_{n=1}^{99} \sqrt{10 - \sqrt{k}} = A\sqrt{2}.$$

Then, we have $A + B = A\sqrt{2}$ and $\frac{B}{A} = \boxed{\sqrt{2} - 1}$ as desired.

□

Problem 11. Scalene $\triangle ABC$ has side lengths $AB = 20$ and $BC = 23$. If the line connecting the incenter and centroid of the triangle is perpendicular to side AB, find the area of $\triangle ABC$.

Answer. $\boxed{20\sqrt{102}}$



Solution. Let G be the centroid and I be the incenter of $\triangle ABC$. Let M be the midpoint of $\overline{AC}$. Let b denote the length of $\overline{AC}$, and s denote the semi-perimeter, equal to $\frac{20+23+b}{2}$. We draw the altitudes from M, G, and C onto line $\overleftrightarrow{AB}$, marking the feet of the altitudes as E, D, and F, respectively.

Now, because $\overleftrightarrow{IG}$ is perpendicular to side $\overline{AB}$, we know that $DB = s - b = \frac{20+23+b}{2} - b = \frac{43-b}{2}$. Additionally, because G divides $\overline{BM}$ into a ratio of 2:1, $DB = \frac{2}{3}EB$.

From here we will find EB. Since E is the foot of the perpendicular from M, $EB = \frac{AB-BF}{2}$. Now, $AB = 20$ from the givens and $BF = 23\cos\angle CBF$ from right $\triangle CFB$. We also know that $\cos\angle CBF = -\cos\angle ABC$ (as they are supplementary).

Let us denote $m\angle ABC = \beta$ for convenience. From Law of Cosines on $\triangle ABC$, we know that $b^2 = 20^2 + 23^2 - 2\cdot 20\cdot 23\cdot\cos\beta$. Thus, $\cos\angle CBF = -\cos\beta = \frac{b^2-20^2-23^2}{2\cdot 20\cdot 23}$. We now have that $BF = 23\cdot\frac{b^2-20^2-23^2}{2\cdot 20\cdot 23}$ and that $EB = \frac{AB-BF}{2} = \frac{20}{2} - 23\cdot\frac{b^2-20^2-23^2}{4\cdot 20\cdot 23}$.

But we also know that $\frac{2}{3}EB = DB = \frac{43-b}{2}$. Thus we get the equation:

$$\frac{2}{3}\left(10 - 23 \cdot \frac{b^2 - 20^2 - 23^2}{4 \cdot 20 \cdot 23}\right) = \frac{43 - b}{2}.$$

This yields the quadratic:

$$-\frac{b^2}{120} + \frac{b}{2} - \frac{851}{120} = 0.$$

This quadratic does have the isosceles triangle condition $b = 23$ as one solution, while Vieta's Formulas tell us that the other solution is $-\frac{\frac{1}{2}}{-\frac{1}{120}} - 23 = 60 - 23 = 37$. Using Heron's Formula to finish, we have:

$$s = \frac{20 + 23 + 37}{2} = 40$$

$$K = \sqrt{40 \cdot (40 - 23) \cdot (40 - 20) \cdot (40 - 37)} = 20\sqrt{102}$$

and the answer is $\boxed{20\sqrt{102}}$ as desired. $\square$

Problem 12. Define the sequence $a_d = d\cdot (d+1)\cdot(d+2)$ for integers $d \geq 1$. Suppose for a positive integer n with $\gcd(n, 6) = 1$, there are 140 values of $k \leq n$ where $\gcd(n, a_k) = 1$. Find the sum of all possible values of n.

Answer. $\boxed{1011}$

Solution. Call a positive integer $k \leq n$ *relatively rhyme* to n if $\gcd(n, a_k) = 1$.
Let $\chi(n)$ be the number of integers that are *relatively rhyme* to n. We aim to prove an explicit formula for $\chi(n)$, when $\gcd(n, 6) = 1$, or, equivalently, $2, 3 \nmid n$.

We want to find how many positive integers k there are such that, for every prime p dividing n, $p \nmid a_k = k(k+1)(k+2)$, or, equivalently, $p \nmid k, k+1, k+2$, since $p \geq 5$. We will instead look at the complement. Consider any prime p that divides n. A positive integer $k \leq n$ is not *relatively rhyme* to n if $p \mid k$, $p \mid k+1$, or $p \mid k+2$. Out of every p consecutive integers, there are three values of k that satisfy this, and since $p \mid n$, this means that $\frac{3}{p}$ of the integers between 1 and n inclusive are not *relatively rhyme* to n due to their gcd being divisible by p.

We now claim that, if $n = p_1^{e_1} p_2^{e_2} \cdots p_j^{e_j}$, then

$$\chi(n) = n \cdot \prod_{i=1}^{j} \left(1 - \frac{3}{p_i}\right).$$

This is because each prime eliminates $\frac{3}{p_i}$ of the integers from being *relatively rhyme* to n. That ratio is constant because primes are relative prime to each other, so removing all integers that are $0, \pm 1 \pmod{p_i}$ does not change the density of integers that are $0, \pm 1 \pmod{p}$ for any other prime p.

Note. This is an informal proof similar to that for proving the explicit formula for $\varphi(n)$, Euler's Totient Function. $\chi(n)$ is very similar to $\varphi(n)$, intended to eliminate three values per prime instead of one.

Now that we have $\chi(n) = 140 = 2^2 \cdot 5 \cdot 7$, we find all primes p such that $p - 3 \mid 140$, being $p = 5$, 7, 13, 17, 23, 31, and 73. We casework on the largest prime that divides n.

If $73 \mid n$, $v_{73}(n)$ is clearly 1 and $\chi\left(\frac{n}{73}\right) = 2$, so $n = 5 \cdot 73 = 365$ works.

If $31 \mid n$, $v_{31}(n)$ is clearly 1 and $\chi\left(\frac{n}{31}\right) = 5$, which has no solutions.

If $23 \mid n$, $v_{23}(n)$ is clearly 1 and $\chi\left(\frac{n}{23}\right) = 7$, which has no solutions.

If $17 \mid n$, $v_{17}(n)$ is clearly 1 and $\chi\left(\frac{n}{17}\right) = 10$, which has two solutions: $n = 13 \cdot 17 = 221$, and $n = 5^2 \cdot 17 = 425$.

A quick check shows that 5, 7, or 13 being the largest prime dividing n yields no solutions. Then, our answer is $365 + 221 + 425 = \boxed{1011}$, as desired. $\square$