LEOPARD

Legit Examination On Proof Assessment by Rishabh Das

1 Introduction

Make sure to submit all progress you make on the problems, as they can earn partial credit. All questions require full justification. Write solutions to different problems on **different sheets of paper**. Label the top right of each page you submit with your name, problem number, and "Page x of y", where this is the xth page you're submitting for this problem, out of y pages for the problem. For a geometry problem, make sure to include a clear diagram at the start of your solution. (This will help you out too!) **Some problems have two parts; often the first part is easier than the second.** Good luck!

2 Problems

Problem 1 (5 Points)**.** Let acute triangle ABC have incenter I. Let the B-angle bisector and C-angle bisector intersect the altitude from A at X and Y, respectively. (The incenter is the intersection of the angle bisectors of a triangle.)

(a) Compute the angles of triangle IXY in terms of the angles of triangle ABC.

(b) Prove that AI is tangent to the circumcircle of triangle IXY. (**Hint:** What angle properties must the tangent to a circumcircle at a vertex satisfy?)

Problem 2 (6 Points)**.** Let n be a positive integer, and let $a_0 = n$. Let $a_{k+1} = a_k - \lfloor \sqrt{a_k} \rfloor$ for all nonnegative integers k.

(a) If $n = 10000$, find the smallest k such that $a_k = 0$.

(b) Find the smallest positive integer n such that $a_{100} \neq 0$.

Problem 3 (7 Points)**.** A positive integer n is written on the board. Alice and Bob take turns erasing the current number m from the board and replacing it with $m - a^2$ for some positive integer a, with Alice going first. The first player to write a negative integer loses.

(a) If $n = 10$, who wins this game?

(b) Are there infinitely many values of n for which Bob wins this game?

Problem 4 (8 Points)**.** Let $S_n = \{1, 2, \ldots, n\}$. For which positive integers $n > 1$ do there exist two permutations, π_1 and π_2, from S_n to S_n satisfying

$$\pi_1(k) \equiv k \cdot \pi_2(k) \pmod{n}$$

for all $k \in S_n$?

Problem 5 (9 Points)**.** Point D is chosen on segment BC of scalene triangle ABC. Points X and Y are chosen on AB and AC, respectively, such that $AXDY$ is a parallelogram. Prove that as D varies, the circumcircle of AXY passes through a fixed point $T \neq A$.

3 Solutions

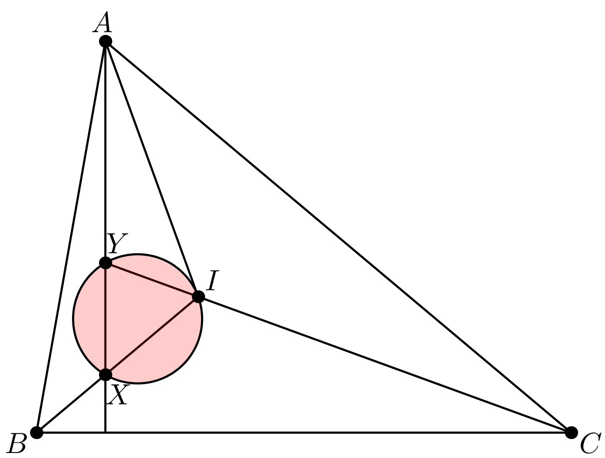
Problem 1

Let acute triangle ABC have incenter I. Let the B-angle bisector and C-angle bisector intersect the altitude from A at X and Y, respectively. (The incenter is the intersection of the angle bisectors of a triangle.)

(a) Compute the angles of triangle IXY in terms of the angles of triangle ABC.

(b) Prove that AI is tangent to the circumcircle of triangle IXY. (**Hint:** What angle properties must the tangent to a circumcircle at a vertex satisfy?)

Solution.



(a) We can compute

$$\angle IYX = 90^\circ - \angle ICB = 90^\circ - \angle C/2.$$

Similarly, $\angle IXY = 90^\circ - \angle B/2$. Since the angles of $\triangle IXY$ must add to 180°, this means $\angle XIY = 90^\circ - \angle A/2$. (Another way to see this is by $\angle BIC = 90^\circ + \angle A/2$.)

(b) In order to prove that AI is tangent to (IXY), we must show $\angle IXY = \angle AIY$. We've already computed $\angle IXY = 90^\circ - \angle B/2$, so we need to show $\angle AIY = 90^\circ - \angle B/2$. We have

$$\angle AIY = 180^\circ - \angle AIC = 180^\circ - (90^\circ + \angle B/2) = 90^\circ - \angle B/2 = \angle IXY,$$

so we're done.

Problem 2

Let n be a positive integer, and let $a_0 = n$. Let $a_{k+1} = a_k - \lfloor \sqrt{a_k} \rfloor$ for all nonnegative integers k.

(a) If $n = 10000$, find the smallest k such that $a_k = 0$.

(b) Find the smallest positive integer n such that $a_{100} \neq 0$.

Solution.

(a) If $a_i = m^2$, then $a_{i+1} = m^2 - m$, and then

$$a_{i+2} = a_{i+1} - (m-1) = m^2 - 2m + 1 = (m-1)^2.$$

This means, as $a_0 = 100^2$, that $a_2 = 99^2$, $a_4 = 98^2$, and so on, until $a_{198} = 1$, and then $a_{199} = 0$. Thus, the smallest k is $k = 199$.

(b) Taking from the previous part, if $a_0 = 50^2$ then it takes 99 steps to reach 0. This means for $a_0 = 50^2 + 50$ that it would take 100 steps.

We claim $a_0 = 50^2 + 50 + 1 = 2551$ that it takes 101 steps, making the answer 2551. We would then have $a_1 = 2501$. Now if $a_i = k^2 + 1$, then $a_{i+1} = k^2 - k + 1$, and then

$$a_{i+2} = a_{i+1} - (k-1) = (k-1)^2 + 1.$$

This means $a_3 = 49^2 + 1$, $a_5 = 48^2 + 1$, and so on, until $a_{99} = 1^2 + 1 = 2$. Then $a_{100} = 1 \neq 0$, and $a_{101} = 0$.

Problem 3

A positive integer n is written on the board. Alice and Bob take turns erasing the current number m from the board and replacing it with $m - a^2$ for some positive integer a, with Alice going first. The first player to write a negative integer loses.

(a) If $n = 10$, who wins this game?

(b) Are there infinitely many values of n for which Bob wins this game?

Solution.

(a) We claim Bob wins this game. If Alice subtracts 1 or 9, then Bob can subtract 9 or 1, respectively, to win the game.

Suppose Alice subtracts 4, so Bob is left with 6. Then Bob can subtract 4, and it's clear to see that from 2, Alice will lose. (Bob can subtract 1 as well to win.)

(b) We claim there are infinitely many values of n for which Bob wins. Suppose otherwise, so N is the largest positive integer for which Bob wins.

Consider $N^2 + N + 1$. Alice can subtract any of $\{1, 4, \ldots, N^2\}$; no matter which number she subtracts, Bob is left with a number at least $N + 1$. However, since we have assumed N is the largest positive integer that's a losing position, this means no matter what number Alice subtracts, it results in a winning position. This means $N^2 + N + 1$ must have actually been a losing position for Alice, contradicting the maximality of N.

Problem 4

Let $S_n = \{1, 2, \ldots, n\}$. For which positive integers $n > 1$ do there exist two permutations, π_1 and π_2, from S_n to S_n satisfying

$$\pi_1(k) \equiv k \cdot \pi_2(k) \pmod{n}$$

for all $k \in S_n$?

Solution. The answer is only $n = 2$. The construction is $\pi_1(k) \equiv \pi_2(k) \equiv k$.

We claim that if $d \mid n$ and $d \mid m$, then $d \mid \pi_2(m)$. Note that if $\pi_2(k)$ is a multiple of d for some k not a multiple of d, then π_1 would take on a value that is a multiple of d at least $\frac{n}{d} + 1$ times: when the input is a multiple of d, and for k. This is a contradiction. Thus, $\pi_2(k)$ being a multiple of d means k is a multiple of d. However, π_2 must be a multiple of d exactly $\frac{n}{d}$ times, so the converse of this is true, which is the desired claim.

Note $\pi_1(0) = \pi_2(0) = 0$ from the above claim. Now, if n is not square-free, then there exists an $m \neq n$ such that $m \mid n$ but $n \mid m^2$. We know $\pi_2(m)$ is a multiple of m, so $\pi_1(m)$ is a multiple of m^2, and is thus a multiple of n, a contradiction. Thus, square-free n certainly fail.

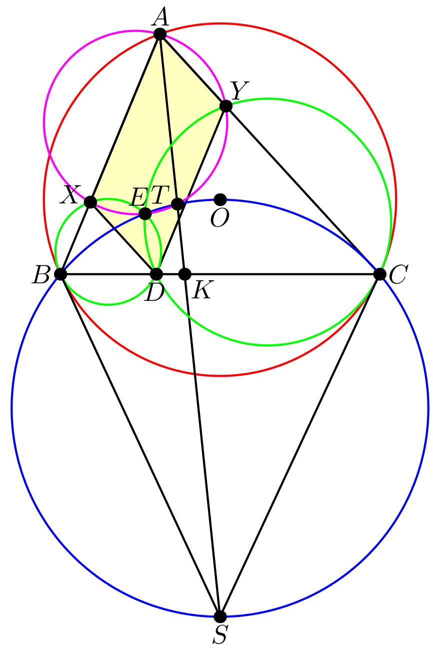
Now suppose n is square-free. Let $n = pm$, where p is prime not equal to 2 and m is relatively prime to p. Then π_2 must map $\{m, 2m, \ldots, (p-1)m\}$ to itself by the claim, and thus π_1 must also do this. But then

$$m^{p-1}(p-1)! \equiv \prod_{k=1}^{p-1} \pi_1(km) \equiv m^{p-1}(p-1)! \prod_{k=1}^{p-1} \pi_2(km) \equiv m^{2(p-1)}((p-1)!)^2 \pmod{n}.$$

However, $m^{p-1} \equiv 1 \pmod{p}$ and $(p-1)! \equiv -1 \pmod{p}$, so this says $-1 \equiv 1 \pmod{p}$, a contradiction.

Problem 5

Point D is chosen on segment BC of scalene triangle ABC. Points X and Y are chosen on AB and AC, respectively, such that $AXDY$ is a parallelogram. Prove that as D varies, the circumcircle of AXY passes through a fixed point $T \neq A$.



We present three solutions, in increasing order of machinery.

Solution 1. Let S be the intersection of the tangents to (ABC) at B and C. Let E be the intersection of (BDX), (CDY), and (AXY) (which exists by Miquel), let K be the intersection of AS and BC, and let T be the intersection of AS and (BOC). We claim T is the fixed point.

We have $\angle BED = \angle DBS = A$ and $\angle CED = A$. This means $\angle BEC = 2A$, so E lies on (BOC). Since ED bisects $\angle BEC$, this means E, D, and S are collinear.

By the shooting lemma, $SD \times SE = SK \times ST$, so $TEDK$ is cyclic. Now

$$\measuredangle ATE = \measuredangle KTE = \measuredangle KDE = \measuredangle BDE = \measuredangle BXE = \measuredangle AXE,$$

so T lies on (AEX), as desired.

Solution 2. Set A to be the origin. Let $X = k\overrightarrow{B}$, and $Y = (1-k)\overrightarrow{C}$, for some real k. Perform a $\sqrt{bc}$-inversion centered at A, followed by a reflection about the A-angle bisector. Then $X' = \frac{1}{k}\overrightarrow{C}$ and $Y' = \frac{1}{1-k}\overrightarrow{B}$. We want to show as k varies, the line $X'Y'$ passes through a fixed point. We claim it always passes through $\overrightarrow{B} + \overrightarrow{C}$. Indeed, we have

$$\overrightarrow{B} + \overrightarrow{C} = kX' + (1-k)Y'$$

is a weighted average of X' and Y', as desired.

Solution 3. Let $f(\bullet)$ denote $\operatorname{Pow}_{(AXY)}(\bullet) - \operatorname{Pow}_{(ABC)}(\bullet)$, which is a linear function in $\bullet$. Let $BD = kBC$ and $CD = (1-k)BC$. Then $f(B) = kc^2$ and $f(C) = (1-k)b^2$. Then, because K is on the A-symmedian,

$$f(K) = \frac{c^2}{b^2+c^2} f(B) + \frac{b^2}{b^2+c^2} f(C) = \frac{b^2c^2}{b^2+c^2}$$

is independent of k. Since $\operatorname{Pow}_{(ABC)}(K)$ is also constant, that means $\operatorname{Pow}_{(AXY)}(K)$ is constant. This means that the line KA must intersect (AXY) at a fixed point, as desired.