NYCMT 2023-2024 Homework #3

NYCMT

October 13 - 20, 2023

These problems are due October 20th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Also write down any progress you make on problems you don't solve. Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.
If you're not going to be present on October 20th, you can scan your solutions and email them to ali40@stuy.edu, dpotievsky40@stuy.edu, and jaeminkim2@hunterschools.org. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or using one of the above emails.
Note that problems are NOT difficulty-ordered, so you should read and try all of them.
Enjoy!

Problem 1. In a best of n series where n is an odd number (with no ties), two players repeatedly play sets until one player has won $\frac{n+1}{2}$ matches, leaving them as the victor and concluding the series. Let $f(n)$ be the probability that this best of n series lasts all n matches, given that each player has a 50% chance of winning each match. Compute the value of $\frac{f(19)}{f(21)}$.

Problem 2. Jaemin has a right circular cone with base radius 3 and height $6\sqrt{2}$. Circle ω is defined as the set of points on the surface of Jaemin's cone that are 4 units away from the tip of the cone, and segment $\overline{AB}$ is a diameter of ω. Andrew the ant is situated at point A and wants to crawl along the surface of the cone to point B. What is the shortest distance Andrew must travel?

Problem 3. If x, y, and z are real numbers such that

$$\sqrt{3z^2 - 2y - 6z - 5} = \sqrt{2x - y^2 + z^2 - 7} - \sqrt{2y - x^2 + z^2 - 12},$$

find $x + y + z$.

Problem 4. Find all positive integer solution pairs (x, y) to the equation:

$$x^3 + 12x - 2 = y^3 + 6y$$

Problem 5. Segment $\overline{AD}$ is a median of $\triangle ABC$ and l is a line through A. E is a point on l such that $CE \parallel AD$. F and G are the feet of the perpendiculars from E and B, respectively, on AD. Prove that $EG \parallel BF$.