Name: __________

Team : __________

1. __________

2. __________

3. __________

4. __________

5. __________

6. __________

7. __________

8. see puzzle

9. __________

10. __________

11. __________

12. __________

– 45 minutes
– no calculators
– integer answers
(except problem 8)

Math Majors of America Tournament for High Schools

2022 Team Round

1. Rectangle $ABCD$ has $AB = 8$ and $BC = 13$. Points P_1 and P_2 lie on AB and CD with $P_1P_2 \parallel BC$. Points Q_1 and Q_2 lie on BC and DA with $Q_1Q_2 \parallel AB$. Find the area of quadrilateral $P_1Q_1P_2Q_2$.

2. How many ways are there to fill in a three by three grid of cells with 0's and 2's, one number in each cell, such that each two by two contiguous subgrid contains exactly three 2's and one 0?

3. There are 522 people at a beach, each of whom owns a cat, a dog, both, or neither. If 20 percent of cat-owners also own a dog, 70 percent of dog-owners do not own a cat, and 50 percent of people who don't own a cat also don't own a dog, how many people own neither type of pet?

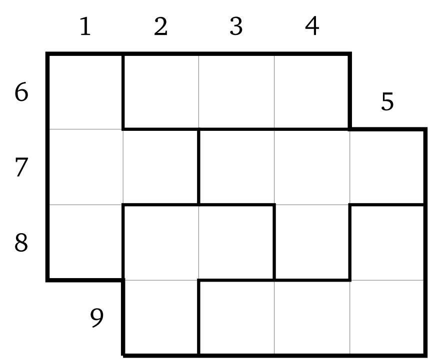
4. How many ways are there to choose three digits A, B, C with $1 \le A \le 9$ and $0 \le B, C \le 9$ such that $\overline{ABC}_b$ is even for all choices of base b with $b \ge 10$?

5. Equilateral triangle $\triangle ABC$ has side length 6. Points D and E lie on $\overline{BC}$ such that $BD = CE$ and B, D, E, C are collinear in that order. Points F and G lie on $\overline{AB}$ such that $\overline{FD} \perp \overline{BC}$, and $GF = GA$. If the minimum possible value of the sum of the areas of $\triangle BFD$ and $\triangle DGE$ can be expressed as $\frac{a\sqrt{b}}{c}$ for positive integers a, b, c with $\gcd(a, c) = 1$ and b squarefree, find $a + b + c$.

6. Prair writes the letters A, B, C, D, and E such that neither vowel are written first, and they are not adjacent; such that there exists at least one pair of adjacent consonants; and such that exactly five pairs of letters are in alphabetical order. How many possible ways could Prair have ordered the letters?

7. $\triangle ABC$ satisfies $AB = 16$, $BC = 30$, and $\angle ABC = 90^\circ$. On the circumcircle of $\triangle ABC$, let P be the midpoint of arc $\widehat{AC}$ not containing B, and let X and Y lie on lines AB and BC, respectively, with $PX \perp AB$ and $PY \perp BC$. Find XY^2.

8. In the number puzzle below, each cell contains a digit, each cell in the same bolded region has the same digit, and cells in different bolded regions have different digits. The answers to the clues are to be read as three-, four-, or five-digit numbers. Find the unique solution to the puzzle, given that no answer to any clue has a leading 0.



DOWN

2. 1 less than a multiple of 13
3. Sum of digits is at least 25
5. Positive difference of this number and 1-Down is divisible by 9

ACROSS

9. Divisible by 19

9. Let f be a monic cubic polynomial such that the sum of the coefficients of f is 5 and such that the sum of the roots of f is 1. Find the absolute value of the sum of the cubes of the roots of f.

10. Suppose that $A_1A_2A_3$ is a triangle with $A_1A_2 = 16$ and $A_1A_3 = A_2A_3 = 10$. For each integer $n \ge 4$, set A_n to be the circumcenter of triangle $A_{n-1}A_{n-2}A_{n-3}$. There exists a unique point Z lying in the interiors of the circumcircles of triangles $A_kA_{k+1}A_{k+2}$ for all integers $k \ge 1$. If $ZA_1^2 + ZA_2^2 + ZA_3^2 + ZA_4^2$ can be expressed as $\frac{a}{b}$ for positive integers a, b with $\gcd(a, b) = 1$, find $a + b$.

11. Every time Josh and Ron tap their screens, one of three emojis appears, each with equal probability: barbecue, bacon, or burger. Josh taps his screen until he gets a sequence of barbecue, bacon, and burger consecutively (in that specific order.) Ron taps his screen until he gets a sequence of three bacons in a row. Let M and N be the expected number of times Josh and Ron tap their screens, respectively. What is $|M - N|$?

12. Let triangle ABC with incenter I satisfy $AB = 3$, $AC = 4$, and $BC = 5$. Suppose that D and E lie on AB and AC, respectively, such that D, I, and E are collinear and $DE \perp AI$. Points P and Q lie on side BC such that $IP = BP$ and $IQ = CQ$, and lines DP and EQ meet at S. Compute SI^2.