NYCMT 2023-2024 Homework #1

NYCMT

September 22 - 29, 2023

These problems are due Friday, September 29th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Also, write down any progress you make on problems you don't solve. Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.
If you're not going to be present on September 29th, you can scan your solutions and email them to ali40@stuy.edu, dpotievsky40@stuy.edu, and jaeminkim2@hunterschools.org. If you will be there, just hand in your responses on paper. If you have any questions, feel free to ask one of us on Discord or using one of the above emails.
Note that problems are NOT difficulty-ordered, so you should read and try all of them. Enjoy!

Problem 1. If t, x, y, and z are real numbers such that

$$\frac{t}{x+y+z} = \frac{x}{y+z+t} = \frac{y}{z+t+x} = \frac{z}{t+x+y},$$

find all possible values of

$$\frac{x+y}{z+t} + \frac{y+z}{t+x} + \frac{z+t}{x+y} + \frac{t+x}{y+z}.$$

Problem 2. A Jaemin-sequence is an infinite sequence of positive integers defined by the first two terms j_1 and j_2, where, for $n > 2$, $j_n = j_{n-1} \cdot j_{n-2}$. Determine the number of distinct Jaemin-sequences where there is at least one $i > 2$ such that $j_i = 46656$.

Problem 3. Find

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}.$$

Problems 4 and 5 are on the back (second page).

Problem 4. Let ABC be a triangle with $BC = 14$ and $\angle A = 120°$. Let its incenter be I. Let $\overline{AI}$ intersect $\overline{BC}$ at D, $\overline{BI}$ intersect $\overline{AC}$ at E, and $\overline{CI}$ intersect $\overline{AB}$ at F. Given $AF = \frac{5}{2}$, $AE = 3$, find ID.

Problem 5. Prove that the simultaneous system of equations

$$a^2 + 2b = 19$$

$$b^2 + 2c = 9$$

$$c^2 + 2a = 8$$

has no solutions in integers.