New York City Team Contest: Solutions

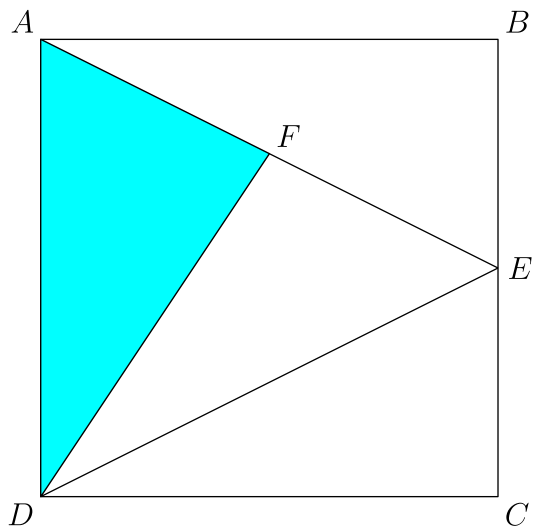
Winter 2022

1. **[6]** Let r and s be the zeroes of $f(x) = x^2 - 7x + 10$. Compute $|\frac{r+s}{r-s}|$.

   Solution: We can factor $f(x) = (x-5)(x-2)$. So, the sum of the zeroes, $r + s = 7$. The positive difference between the zeroes, $|r - s| = 3$. So, $|\frac{r+s}{r-s}| = \frac{7}{3}$.

2. **[6]** Let $ABCD$ be a square with side length 1. Let E be the midpoint of side BC, and let F be the midpoint of segment AE. Compute the area of $\triangle ADF$.

   Solution:



   Note that the area of triangle $\triangle ADE$ is $\frac{1}{2}1(1) = \frac{1}{2}$ since both the base and height of $\triangle ADE$ are 1.

   Additionally, since F is a midpoint of AE, line DF splits $\triangle ADE$ into two triangles of equal area, $\triangle AFD$ and $\triangle DFE$, so the area of $\triangle AFD = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$

3. **[7]** Compute

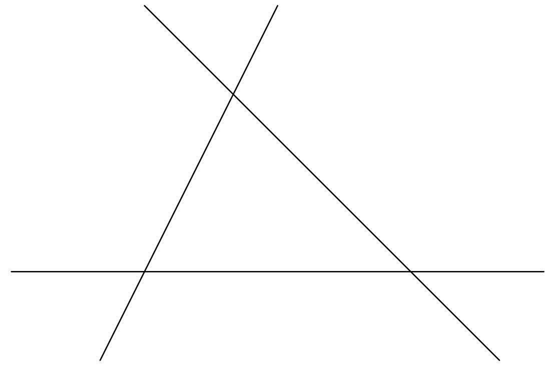
$$(123 + 231 - 321)\left(1^{(2^3)} + 2^{(3^1)} - 3^{(2^1)}\right)\left(\frac{12}{3} + \frac{23}{1} - \frac{32}{1}\right)$$

   Solution: Note that the middle term can be computed as $1 + 8 - 9 = 0$, implying that the final product is 0 as well.

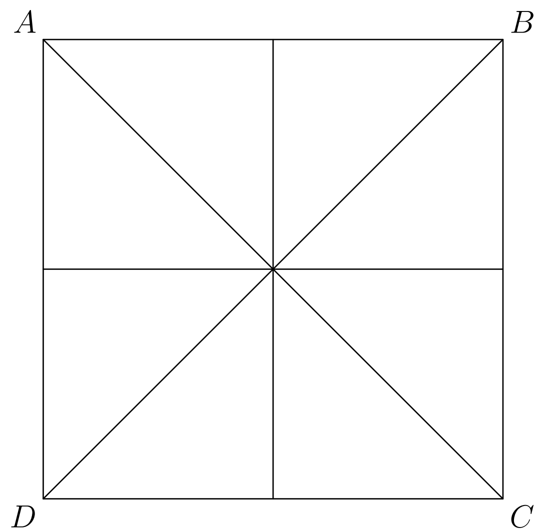
4. **[7]** Compute the least number of line segments needed to dissect a square into 8 regions, all of which are triangles.

   Solution: We claim that we need at least 4 segments.

   3 lines can split the plane into at most 7 parts:



Then, there is no way that 3 segments can split the square into 8 parts, let alone triangles. We can construct 4 segments such that the condition is satisfied:



4 segments are needed and 4 segments are enough, so the answer is 4.

5. **[8]** Julius Randle is given the ball during the waning seconds of a New York Knicks basketball game. The probability that he scores and wins the game is $\frac{1}{10}$. The probability that he turns the ball over and loses the game is $\frac{1}{10}$. The probability that he misses the shot but doesn't lose the game is $\frac{4}{5}$, repeating this in overtime. This continues until either Julius hits the shot, or loses the game. What is the probability that Julius hits the shot and wins the game for the Knicks?

   Solution: Note that this system is symmetric, since the probability that Julius wins the Knicks the game on any given stage is equal to the probability that he loses the game at that same stage. Since the game can't end in a tie, the probability is simply $\frac{1}{2}$.

6. **[8]** Let $s(n)$ be the sum of the digits of n. Compute the sum of the 2 smallest positive integers n such that $n = 6 \cdot s(n) - 1$.

   Solution: First, note that no 1 digit numbers work. Now, let $n = 10a + b$, where a and b are the digits of the two-digit number n. The given equation rewrites as $10 + b = 6(a + b) - 1 \implies 4a + 1 = 5b$. Testing values of b yields $b = 1 \implies a = 1$ and $b = 5 \implies a = 6$, giving the two smallest values of n to be 11 and 65, respectively. This gives a sum of 76 as desired.

7. **[9]** Jerry colors some set of unit lattice squares (squares of side length 1 with vertices on lattice points) purple such that any two lattice squares can be connected by a path of purple squares that touch along an edge. Given that the smallest rectangle that can be constructed with sides perpendicular to the axes containing all of the colored squares has area 120, find the smallest number of squares Jerry could have colored.

Solution: We claim that if the smallest box containing a path of purple squares has dimensions $A \times B$, there are at least $A + B - 1$ purple squares. To show this, label the rows of the box 1 to A from bottom to top, and label the columns of the box 1 to B from left to right. Because there is a square in row 1 and a square in row A, we need at least $A - 1$ vertical steps in our path. Similarly, we need at least $B - 1$ horizontal steps in our path. Thus, there are at least $A + B - 2$ steps in our path, which means that the path contains at least $A + B - 1$ purple squares, as desired.

Let the dimensions of our bounding box be $A \times B$. We know that $AB = 120$, and A, B are integers, so the smallest possible value of $A + B - 1$ is 21. By the claim above, we need at least 21 purple squares. We note that 21 is achievable if we create an L-shaped path that starts at any square, goes 11 steps right then 9 steps up.

8. **[9]** Compute the number of ways to fill crates numbered 1 though 8 with purple pandas, teal terrapins, or grey grapes, such that each crate has exactly one object, and any crate filled with grapes is labeled with an odd number.

Solution: In each of the crates labelled $2, 4, 6, 8$, we have exactly 2 options to place an object in them: either a panda or a terrapin. In each of the crates labelled $1, 3, 5, 7$, we can place any of the 3 objects into each crate. Then the final answer is $2^4 \cdot 3^4 = 16 \cdot 81 = 1296$.

9. **[10]** John is mashing buttons on his calculator. His first button is an integer between 1 and 9; his second button is one of the symbols $+, -, \times, \div$; his third is an integer between 1 and 9. What is the probability that his button-mashed expression results in a positive integer?

Solution: Note that there are $9 \cdot 4 \cdot 9 = 324$ total ways for John to create an expression. If the operation he chooses in the second slot is either $+$ or $\times$, then all such pairs of integers chosen in the first and third slots work, giving $9 \cdot 9 = 81$ in each case, for 162 so far.

If the operation that John chooses is subtraction, then we require that the first integer John chooses is larger than the second; there are exactly $\binom{9}{2}$ ways to choose 2 distinct integers from 1 to 9, and after choosing these 2 integers, the larger one must appear first, so each choice of 2 integers results in exactly one valid subtraction. Then there are $\binom{9}{2} = 36$ choices of the integers that are valid in this case.

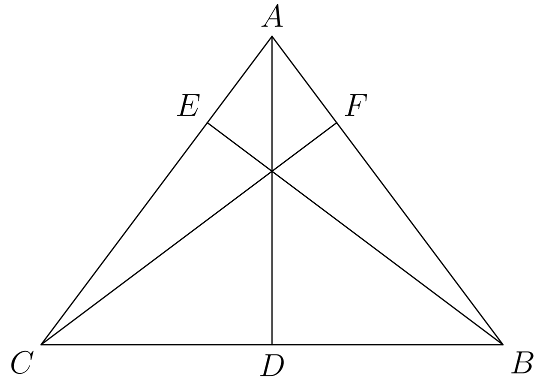
Finally, if John chooses to divide the two integers, then we need the first to be a multiple of the second. We now casework on what the second integer can be.
If it is 1, then any of the 9 integers can be the first one chosen. If it is 2, then we need to choose one of the 4 even integers in the interval. If it is 3, then we need to choose one of the 3 multiples of 3. If it is 4, then either 4 or 8, and if it is 5 or above, then the only valid integer to choose is itself. This gives $9 + 4 + 3 + 2 + 1 + 1 + 1 + 1 + 1 = 23$ choices for the 2 integers in this case.

Then overall, there are $162 + 36 + 23 = 221$ total expressions that result in a positive integer, and so the probability is $\frac{221}{324}$.

10. **[Up to 10]** Submit two points $A : (x_1, y_1), B : (x_2, y_2)$ such that $0 \leq x_1, x_2, y_1, y_2 \leq 5\sqrt{2}$. Let L be the length of your line segment $\overline{AB}$, and let N be the number of intersections between your line segment and the segments of other teams. If your segment contains or is entirely contained within another team's segment, you will receive 0 points. Otherwise, you will receive $\dfrac{L}{N+1}$ points.

11. **[11]** Let $\triangle ABC$ be a triangle with $BC = 5$ and $h_B = h_C = 4$, where h_B and h_C represent the heights from B to side AC and C to side AB, respectively. Compute the area of $\triangle ABC$.

Solution:



Let D, E, F be the feet of the altitudes from A, B, C to BC, CA, AB respectively.

The statement of the problem tells us that $BE = CF = 4$. Applying the Pythagorean Theorem on $\triangle CEB$ and $\triangle CFB$, we get that $CE = FB = 3$.

Note that $\triangle ACD$ and $\triangle BEC$ are similar since they both share angle $\angle ACB$ and have a right angle. Thus $\frac{AD}{DC} = \frac{BE}{EC} = \frac{4}{3}$.

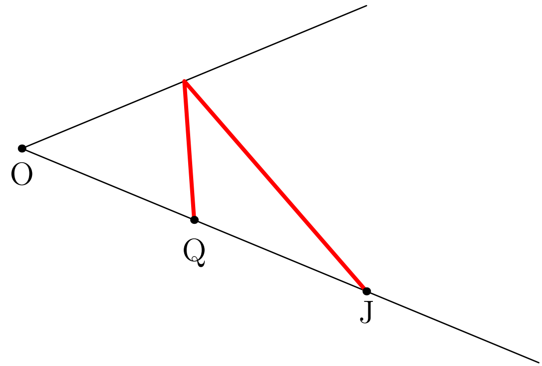
However, since $BE = CF$, we can conclude that $AC = AB$ since $\frac{1}{2}AC(BE) = [ABC] = \frac{1}{2}AB(CF)$. Thus D is actually a midpoint of BC, so $DC = \frac{5}{2}$.
Substituting this into $\frac{AD}{DC} = \frac{4}{3}$ we get $AD = \frac{10}{3}$, so the area of triangle ABC is $\frac{1}{2}\left(\frac{10}{3}\right)5 = \frac{25}{3}$.

12. **[11]** Rishbah has a set of 13 cards, numbered 1 through 13. He chooses a hand of 7 cards. Given that there are at least 5 consecutive numbers among the chosen cards, compute the number of hands Rishbah could have chosen.

Solution: Call a set of k consecutive cards a k-straight. For any straight of exactly 5 cards, there are $\binom{8}{2}$ ways to choose the other 2 cards to form a set of 7 cards with a straight. There are 9 possible straights (a straight can start with any card from 1 to 9). Thus there are $9 \cdot \binom{8}{2} = 252$ sets that we've counted so far. However, we've overcounted cases where our straight connects to one of the 2 cards we've chosen: 6-straights and 7-straights. We've counted each set with a 6-straight twice, and each set with a 7-straight three times. Note that if we casework on the straight of length at least 6, then we will subtract out sets with a straight of length exactly 6 once, and we will subtract out sets with a straight of length 7 twice, which will leave us counting each set once. Then for each straight of length 6, there are 7 options for the last card. There are 8 possible straights of length exactly 6, which means that we must subtract out $7 \cdot 8 = 56$ from our original count. Then the total number of sets that satisfy that 5 cards are consecutive is $252 - 56 = 196$ as desired.

13. **[12]** Jerry is playing a game of pool on a very strange table. The table is an isosceles triangle; the two legs form a 30° angle. Jerry hits a ball that is on one of the legs of the triangle. The ball bounces off of the other leg before hitting the first wall halfway between its original position and the vertex of the triangle. Compute the acute angle formed between the wall and the path of the ball at the starting point. (Note that the acute angle formed between the ball's path and the wall is the same before and after colliding with the wall.)

Solution: Let O denote the corner, J denote the point where Jerry is standing, P be the point where the pool ball hits the wall, and Q be the midpoint of OJ. Let Q' be the reflection of Q over the reflective wall. Then JPQ' are collinear. We also have $\angle POQ' = \angle QOQ' = 2\angle QOP = 60$. Also, we have $OQ' = OQ = OP/2$. This implies that OPQ' is a 30-60-90 triangle, so $\angle OPQ = \angle OPQ' = 30$.

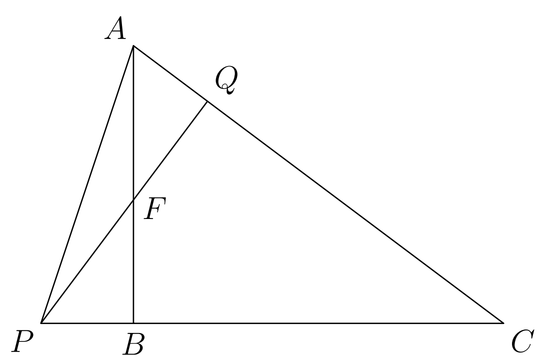


14. **[12]** A figure is constructed recursively. At time $t = 0$, an equilateral triangle of side length 1 is drawn. Each minute after, an equilateral triangle of side length 1 is constructed on each side of an already existing equilateral triangle of side length 1. For example, at time $t = 2$ minutes, there are 10 equilateral triangles of side length 1 drawn. Compute the number of equilateral triangles of side length 1 drawn after 100 minutes.

    Solution: Let $\triangle ABC$ be the first triangle, with B to the right of A and C below segment AB. Call a triangle we draw nice if it is below line BC and above line AC. Let x_n be the number of triangles after n minutes. Let y_n be the number of nice triangles after n minutes. By rotational symmetry about triangle ABC, $x_n = 3y_n + 1$. It suffices to compute y_{100}. Consider the set S_n of nice triangles after n minutes. We see that if n is odd then S_n consists of $(n+1)/2$ down-right "rows" of n triangles each, and if n is even then it consists of $n/2$ up-right "columns" of $n+1$ triangles each. In either case, $y_n = |S_n| = n(n+1)/2$, giving $x_{100} = 1 + 3 \cdot 100 \cdot 101/2 = 15151$.

15. **[13]** Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $\angle ABC = 90°$. Let P be a point on line BC, and let Q be its projection onto line AC. If Q lies between A and C, and $\angle APQ = \frac{1}{2}\angle ACB$, compute AQ.

    Solution:



    Let F be the intersection of PQ and AB.

    Let $x = \angle ACB$, then $\angle BAC = 90 - x$, $\angle APQ = \frac{x}{2}$ and $\angle QAP = 90 - \frac{x}{2}$.
    Thus $\angle BAP = \angle QAP - \angle BAC = (90 - \frac{x}{2}) - (90 - x) = \frac{x}{2}$.

    This tells us that $\triangle APF$ is isosceles, or that $AF = FP$. It also tells us that $\triangle ABP$ and $\triangle PQA$ are congruent, so $\angle APB = \angle PAQ$, meaning that $AC = CP = 5$.

    Since $\triangle AQP$ and $\triangle PBA$ are congruent, $AQ = PB = PC - BC = 5 - 4 = 1$.

16. **[13]** Find the sum of all positive integers n such that $n + 4$ divides $4n^2 + 24n + 46$.

    Solution: If $n + 4$ divides $4n^2 + 24n + 46$, $\frac{4n^2+24n+46}{n+4}$ must be an integer. Performing polynomial division, we get that $4n + 8 + \frac{14}{n+4}$ must be an integer. $4n + 8$ is always an integer, so we want $n + 4$ to divide 14. Since $n > 0$, $n + 4$ equals either 7 or 14, making $n \in \{3, 11\}$. Taking the sum of all possibilities gives $3 + 11 = 14$.

17. **[14]** Let $P(x) = x^4 - 8x^3 + 21x^2 - 14x + 6$ have roots r, s, t, u. Compute $(2 + r^2)(2 + s^2)(2 + t^2)(2 + u^2)$.

Solution: Note that since $P(x)$ is monic, we can say that $P(x) = (x - r)(x - s)(x - t)(x - u)$.
If we multiply each term of the product by -1, we preserve the final value of the product (since $(-1)^4 = 1$). Then we wish to compute $(-2 - r^2)(-2 - s^2)(-2 - t^2)(-2 - u^2)$. Using difference of squares, we want

$$(\sqrt{-2} + r)(\sqrt{-2} - r)(\sqrt{-2} + s)(\sqrt{-2} - s)(\sqrt{-2} + t)(\sqrt{-2} - t)(\sqrt{-2} + u)(\sqrt{-2} - u)$$

Focus on $(\sqrt{-2} - r)(\sqrt{-2} - s)(\sqrt{-2} - t)(\sqrt{-2} - u)$. This is simply $P(\sqrt{-2}) = -32 + 2\sqrt{-2}$. For the product $(\sqrt{-2} + r)(\sqrt{-2} + s)(\sqrt{-2} + t)(\sqrt{-2} + u)$, again note that we can negate each term in the product, so we wish to compute $(-\sqrt{-2} - r)(-\sqrt{-2} - s)(-\sqrt{-2} - t)(-\sqrt{-2} - u) = P(-\sqrt{-2}) = -32 - 2\sqrt{-2}$. Then our original product can be expressed as $P(\sqrt{-2}) \cdot P(-\sqrt{-2}) = (-32 + 2\sqrt{-2})(-32 - 2\sqrt{-2}) = (-32)^2 - (2\sqrt{-2})^2 = 1024 - (-8) = 1032$, and we are done.

18. **[14]** Compute $(14!)^2$ mod 31.

Solution: We rewrite the expression as

$$(1 \cdot 2 \cdot 3 \cdots 14)(1 \cdot 2 \cdot 3 \cdots 14) \equiv (1 \cdot 2 \cdot 3 \cdots 14)(-1 \cdot -2 \cdot -3 \cdots -14)(-1)^{14} \equiv (1 \cdot 2 \cdot 3 \cdots 14)(30 \cdot 29 \cdot 28 \cdots 17) \bmod 31$$

Let this equal k mod 31. Now using Wilson's theorem, we know that $30! = 30$ mod 31. Then

$$15 \cdot 16 \cdot k \equiv (1 \cdot 2 \cdot 3 \cdots 14)(30 \cdot 29 \cdot 28 \cdots 17) \cdot 16 \cdot 15 \equiv 30 \bmod 31$$

$$\implies 15 \cdot 16 \cdot k = 30 \bmod 31$$

Dividing by 15 on both sides yields $16k \equiv 2 \equiv 64$ mod 31, implying $k = 4$ as desired.

19. **[15]** How many lattice paths from $(0, 0)$ to $(6, 6)$ do not touch the lines $y = x - \pi$ or $y = x + \pi$?

Solution: Note that there does not exist a path that intersects both of these lines, since the region of points below the first line and in the square with vertices $(0, 0), (0, 6), (6, 6), (6, 0)$ have $x \geq \pi$, and the region of points below the first line and in the same square have $x \leq 6 - \pi$, and no value of x satisfies both of these. Thus it suffices to compute the total number of paths that intersect the line $y = x + \pi$, and subtract twice this value from the total number of paths.

To count the number of paths that cross through this line, we make use of an argument analogous to that of the Catalan numbers. Given a path that crosses the line for the first time at $(a, a + \pi)$, consider swapping each move after reaching the point $(a, a + 4)$. Since the number of up moves and the number of right moves to get to $(a, a + 4)$ differ by 4, then the number of right moves and the number of up moves to get from $(a, a + 4)$ to $(6, 6)$ also differ by 4. Swapping these gives that at the end, our new path will have a number of up moves that is 8 greater than the number of right moves, and it will have 12 total moves, implying that the new path will end up at the point $(2, 10)$. Reversing this argument (by considering paths that go to $(2, 10)$ and swapping moves after they first intersect the line $y = x + \pi$) shows that this is a bijection, so the number of paths that cross through the line $y = x + \pi$ is exactly equal to the number of paths to $(2, 10)$, of which there are $\binom{12}{2} = 66$.

Finally, there are $\binom{12}{6} = 924$ total paths, and we subtract out $2 \cdot 66 = 132$ bad paths to get 792 paths that don't cross either line.

20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.

(1) For all functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, if there exists integers m and n such that $f^m(x) = x$ and $g^n(x) = x$, then there exists an integer k such that $(f \circ g)^k(x) = x$.

Solution: We can find a construction such that $f^2(x) = x$ and $g^2(x) = x$ ($m = n = 2$), but there is no such k for which $(f \circ g)^k(x) = x$:

$$f(x) = \begin{cases} x+1 & \text{if } x \in \{2n+1 : n \in \mathbb{Z}\} \\ x-1 & \text{if } x \in \{2n : n \in \mathbb{Z}\} \\ x & \text{if } x \notin \mathbb{Z} \end{cases} \tag{1}$$

and

$$g(x) = \begin{cases} x-1 & \text{if } x \in \{2n+1 : n \in \mathbb{Z}\} \\ x+1 & \text{if } x \in \{2n : n \in \mathbb{Z}\} \\ x & \text{if } x \notin \mathbb{Z} \end{cases} \tag{2}$$

Then $f^2(x) = x$ and $g^2(x) = x$, but

$$(f \circ g)(x) = \begin{cases} x-2 & \text{if } x \in \{2n+1 : n \in \mathbb{Z}\} \\ x+2 & \text{if } x \in \{2n : n \in \mathbb{Z}\} \\ x & \text{if } x \notin \mathbb{Z} \end{cases} \tag{3}$$

If x is an odd integer, $(f \circ g)(x)$ increases it by 2. If we do this many times, it keeps increasing, so $(f \circ g)^k(x)$ cannot possibly be x for any k, and so we have a valid counterexample. The statement is False.

(2) The sum of any finite arithmetic sequence of positive integers with at least three terms is composite.

Solution: Let the arithmetic sequence have length d. If $d = 2k \geq 4$ is even, then each pair of opposite terms sums to a fixed number n, so the total sum is equal to $\frac{2kn}{d} = kn$, where both k and n are at least 2, so this always yields a composite sum.

If $d = 2k - 1 \geq$ is odd, then each pair of opposite terms (excluding the middle term!) sums to twice the middle term. Let the middle term be n. Then the total sum is $(k-1)(2n) + n = (2k-1)(n)$, where again both factors are at least 2. This also gives a composite sum, so in all cases, the sum is composite.

(3) There are multiple functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + 2y) = f(x) + f(y)$.

Solution: Firstly, plugging in $x = y = 0$ yields $f(0) = 2f(0) \implies f(0) = 0$. Now, Let $x + 2y = 0$, i.e. $x = -2y$. Then $0 = f(0) = f(x)+f(y) = f(y)+f(x) = f(y+2x) = f(y+(-2y)) = f(-y)$. Then for all y, $f(-y) = 0$, implying $f(a) = 0$ for all $a \in \mathbb{R}$. Then there is only one function that satisfies this property: the zero function, so the statement is false.

(4) Every triangle with inradius 1 and integer side lengths also has a right angle.

Solution: The claim is that the only triangle with inradius 1 and integer side lengths is a 3-4-5 triangle, which has a right angle. Let a, b, c be the triangle's side lengths, and let $s = \frac{a+b+c}{2}$ be the triangle's semiperimeter. Let $x = s - a$, $y = s - b$, $z = s - c$. If s is odd, then x, y, z must all be half of odd integers, while if s is even, then x, y, z are all integers. By Heron's, we must have:

$$\sqrt{xyz(x+y+z)} = sr = x + y + z \implies xyz = x + y + z.$$

At this point we can see that x, y, z must all be integers, because if they are all half integers then xyz would have an 8 in the denominator while $x + y + z$ would have a 2 in the denominator. We can assume WLOG that $x \geq y \geq z$. If $z \geq 2$, then $xyz \geq 4x > x + x + x \geq x + y + z$. Thus, we must have that $z = 1$. This gives us

$$xy = x + y + 1 \implies (x-1)(y-1) = 2.$$

This forces $x = 3,\ y = 2,\ z = 1$, which gives $a = 3,\ b = 4\ c = 5$, as desired.

(5) Let $f(a)$ be a function that returns the sum of all integers b between 1 and a inclusive such that the gcd of a and b is 1. Then there exist positive integers m and n such that $f(m)f(n) = f(mn)$.

Solution: The claim is that $f(a) = \frac{a\varphi(a)}{2}$. We can see this by noting that if b is relatively prime to a, then so is $a - b$. Thus if we look at $2f(a)$ we get

$$\begin{aligned} 2f(a) &= \sum_b b + \sum_b (a - b) \\ &= \sum_b a \end{aligned}$$

where the sums run over all b from 1 to a inclusive such that a and b have a GCD of 1.
However, looking at $\sum_b a$, we can see that this should be equal to ax, where x is the number of possible values of b. Recalling that $\varphi(a)$ counts the number of b that are relatively prime to a and at most a, we can see that $x = \varphi(a)$, so

$$\begin{aligned} 2f(a) &= a\varphi(a) \\ f(a) &= \frac{a\varphi(a)}{2} \end{aligned}$$

Now, we want to determine if there exist positive integers m, n such that

$$\frac{m\varphi(m)}{2}\frac{n\varphi(n)}{2} = \frac{mn\varphi(mn)}{2}$$

This rewrites to finding positive integers m, n such that

$$\varphi(m)\varphi(n) = 2\varphi(mn)$$

However $\varphi(mn)$ is at least as big as $\varphi(m)\varphi(n)$. This is because any integers $1 \le c \le m$ and $1 \le d \le n$ such that c and m have a GCD of 1 and d and n have a GCD of 1 corresponds to at least one integer $1 \le e \le mn$ such that e and mn have a GCD of 1. Similarly every integer $1 \le e \le mn$ such that e and mn have a GCD of 1 corresponds to exactly one $c \equiv e \mod m$ and $d \equiv e \mod n$.

Thus, $\varphi(mn) \ge \varphi(m)\varphi(n)$ so $2\varphi(mn) > \varphi(m)\varphi(n)$ so there cannot be integers satisfying the desired equality.

(6) For all positive integers n, and all odd prime factors p of $n^8 + 1$, we have $16 | p - 1$.

Solution: We claim that this is true. Assume for contradiction that $p - 1$ is not divisible by 16, and let $g = \gcd(p - 1, 16)$. We know that g must be a factor of 8. Clearly, p does not divide n, so by Fermat's Little Theorem we have $n^{p-1} - 1$ is divisible by p. We also know that p divides $n^{16} - 1 = (n^8 + 1)(n^8 - 1)$. Thus, we have $p | \gcd(n^{p-1} - 1, n^{16} - 1) = n^g - 1$. Because g divides 8, $n^g - 1$ divides $n^8 - 1$, so we have that $p | n^8 - 1$. This means that $p | (n^8 + 1) - (n^8 - 1) = 2$, which means $p = 2$. This contradicts the assumption that p is an odd prime. Thus, we must have that $16 | p - 1$.

21. **[16]** A single strip of paper has the numbers $1, 2, \ldots, 10$ written on it in that order. Every minute, Max makes a cut between some two integers on a strip of paper that contains at least two of the ten integers. He then moves any piece with exactly one integer cut during that minute to the leftmost side, maintaining the order they followed after cutting. He stops once every piece has one integer on it. Find the total number of ways Max can cut the strips so that 8 is the first integer from the left when he stops.

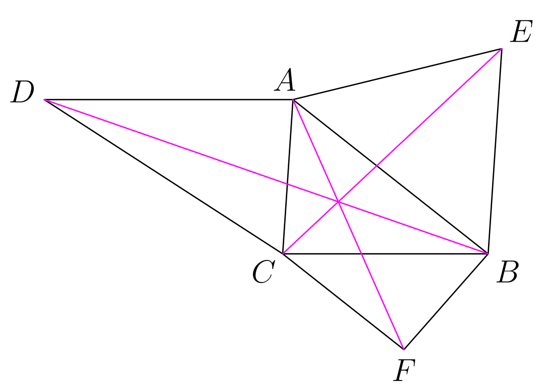
Solution: We claim that it is necessary and sufficient to make the cut between the 8 and 9 last. Note that there will be 9 cuts made in total: the cut between the numbers 1 and 2, the cut between the numbers 2 and 3, and so on until the cut between 9 and 10. If the cut between the 8 and 9 is performed last, then all other cuts have been performed, so each number is already on an individual piece of paper and has been moved to the left. Then the cut between the 8 and 9 translates these to the start, and so the 8 is the start of the string, so the $8-9$ cut being last is sufficient. In a similar manner, if the $8-9$ cut were NOT last, then some other cut is last: say this is the cut between the numbers k and $k+1$. Since this is the last of the 9 cuts, all other numbers are already on individual strips of paper and have already been moved to the left. Then the cut between k and $k+1$ translates both k and $k+1$ to the left, so the final string will start with $k \neq 8$. Thus it is also necessary for the final cut to be between 8 and 9, so we must count the total number of ways to make 9 cuts such that this cut is last. This is simply $8! = 40320$, since the other 8 cuts can be made in any order, making the answer 40320.

22. **[16]** Let $S = a_1, a_2, a_3, \ldots$ be an infinite increasing sequence of positive integers. We have that any element a_i of the sequence is divisible by either 1071 or 1072, but it is not divisible by 107. Let k_S be the maximum possible value of $a_{i+1} - a_i$ among all i. Compute the smallest possible value of k_S, over all possible S.

Solution: Note that it suffices to look at the increasing sequence $a_1, a_2, a_3, \ldots$, where each a_i is as small as possible. We claim that the answer is $2 \cdot 1071 = 2042$. To see that this is a lower bound, consider what happens near $N = 1071 \cdot 1072 \cdot 107$. The largest a_i in this sequence that is less than N is $1071 \cdot (1072 \cdot 107 - 1)$, and the smallest a_i in this sequence that is larger than N is $1071 \cdot (1072 \cdot 107 + 1)$. These have difference $1071 \cdot 2$ as desired. To see that this is an upper bound, we can focus solely on the multiples of 1071 that appear in this sequence. Consider any 3 multiples of 1071; at most one of them is a multiple of 107. Then if these are $1071k, 1071(k+1)$, and $1071(k+2)$, then the maximum possible difference between two of these is $1071 \cdot 2 = 2142$. Since 2142 is both at least and at most k_S in this case, 2142 is the answer.

23. **[17]** Let $\triangle ABC$ be a triangle with $AB = 4, BC = 6, CA = 5$. Let D be the intersection of the B-angle bisector and the line through A parallel to BC. Let E be the intersection of the C-angle bisector and the line through B parallel to AC. Let F be the intersection of the A-angle bisector and the line through C parallel to BA. Compute the area of (non convex) hexagon $AEBFCD$.

Solution: Note that $\angle ABD = \angle DBC = \angle ADB$ by the line DB being an angle bisector, and AD being parallel to BC. Then $\triangle ADC$ is isosceles, so $AD = 4$. Similarly, $BE = 6$ and $CF = 5$. Now, we compute the ratio of the area of $AEBFCD$ to the area of $\triangle ABC$. We split $AEBFCD$ into $\triangle ADC, \triangle BEA, \triangle CFB$, and $\triangle ABC$. Note that $\triangle ADC$ and $\triangle ABC$ have the same height (from A to BC and C to AD), so the ratio of their areas is simply the ratio of AD to BC, which is $\frac{4}{6}$. Similarly, the ratio of the areas of $\triangle BEA$ to $\triangle ABC$ is $\frac{6}{5}$, and the ratio of the areas of $\triangle CBF$ to $\triangle ABC$ is $\frac{5}{4}$. Thus the total area of the hexagon is $(1 + \frac{4}{6} + \frac{6}{5} + \frac{5}{4} = \frac{247}{60}$ times that of $\triangle ABC$. Using Heron's formula, we can compute the area of $\triangle ABC$ to be $\frac{15\sqrt{7}}{4}$, and so the final answer is $\frac{247}{60} \cdot \frac{15\sqrt{7}}{4} = \frac{247\sqrt{7}}{16}$.



24. **[17]** Mr. Cocoros and Rishabh are taking a tour of the fourth floor, which has 8 rooms, including rooms 403 and 407. In order to avoid being supsicious, Mr. Cocoros won't take Rishabh to the same room twice on the tour. How many ways are there for Mr. Cocoros to take Rishabh on a tour of the fourth floor that starts in room 403 ends in room 407 without being suspicious?

Solution: Let the other 6 rooms be A, B, C, D, E, F. Since we don't want to visit any room twice, we want to find the number of permutations of some number of these rooms; the entire trip will start from 403, go through the permutation of the rooms (in order), and go from the last room in the permutation to 407. We can count this by casework on the number of rooms.

For $6 \geq n \geq 1$, consider a permutation of n of the rooms. There will be 6 options for the first room, 5 options for the second (unless $n = 1$), and in general there will be $7 - k$ options for the k-th room in the sequence. Then there will be $\prod_{i=1}^{n}(7-i)$ total permutations of length n. Summing this for all n from 1 to 6, we get $6+(6)(5)+(6)(5)(4)+(6)(5)(4)(3)+(6)(5)(4)(3)(2)+(6)(5)(4)(3)(2)(1) = 6+30+120+360+720+720 = 1956$. Adding in the empty permutation (i.e. the tour that goes directly from room 403 to 407), we get 1957 total paths as desired.

25. **[18]** Call a function $f : \mathbb{Z} \to \{0, 1, 2, 3, 4\}$ *1-multiplicative* if for every a,

- $f(a+5) = f(a)$
- There exists at least one value $b \not\equiv a \mod 5$ for which $f(ab) \equiv f(a) \cdot f(b) \mod 5$.

Find the number of 1-multiplicative functions.

Solution: Note that the first condition makes it sufficient to assign values to $f(0), f(1), f(2), f(3), f(4)$. We will also say a pairs with b if a and b satisfy the second condition.

Let us first consider $f(0) \equiv f(0) \cdot f(b) \mod 5$, for $b \not\equiv 0 \mod 5$. Because each value $k, 2k, 3k, 4k, 5k$ will be distinct mod 5 for $k \not\equiv 0 \mod 5$, this can only be true when either $f(0) = 0$ or $f(b) = 1$ is true. When $f(0) = 0$, this holds for all $b \in \{1, 2, 3, 4\}$, meaning every value will have a pairing and we can assign any of the five values to $f(1), f(2), f(3), f(4)$, totaling $5^4 = 625$ ways. In all other cases, $f(0) \neq 0$ and we must have some value $b \not\equiv 0 \mod 5$ for which $f(b) = 1$.

Now let us consider $f(b) = f(1) \cdot f(b)$, $b \not\equiv 1 \mod 5$. This implies $f(1) = 1$ or $f(b) = 0$. Similarly, if $f(1) = 1$, this will pair with all $b \not\equiv 1 \mod 5$, totaling $4 \cdot 5^3 = 500$ ways. Otherwise, we need some $f(b) = 0$, where $b \not\equiv 0, 1 \mod 5$.

These conditions give us that among $f(2), f(3), f(4)$, we need some value equivalent to 0 (pairing with 1) and another equivalent to 1 (pairing with 0). We can notice that if all three values are either 0 or 1 (where there is at least one of each), this function will satisfy, totaling $4 \cdot 4 \cdot 6 = 96$ ways.

Otherwise, the only way we can achieve a value not equal to 0 or 1 among $f(2), f(3), f(4)$ is if some pairing of values $a, b \in \{2, 3, 4\}$ satisfies $ab \equiv 1 \mod 5$. This is only true for $2 \cdot 3 \equiv 1 \mod 5$, which gives us four assignments:

$$f(2) = f(1) = 0 \implies f(3) \in \{2, 3, 4\} \implies 3 \cdot 4 = 12 \text{ ways}$$

$$f(2) = 1 \implies f(3) = f(1) \in \{2, 3, 4\} \implies 3 \cdot 4 = 12 \text{ ways}$$

$$f(3) = f(1) = 0 \implies f(2) \in \{2, 3, 4\} \implies 3 \cdot 4 = 12 \text{ ways}$$

$$f(3) = 1 \implies f(2) = f(1) \in \{2, 3, 4\} \implies 3 \cdot 4 = 12 \text{ ways}$$

The total number of functions is then $625 + 500 + 96 + 48 = 1269$.

26. **[18]** Compute

$$\frac{\sum_{n=1}^{50}(n^2+1) \cdot n!}{50!}$$

Solution: Note that

$$(n-1)\cdot n! + (n^2+1)\cdot n! = (n^2+n)\cdot n! = n\cdot(n+1)! = ((n+1)-1)\cdot(n+1)!$$

Thus, by adding $(1 - 1\cdot 1!)$ (which is equal to 0) to the numerator of the expression we want to compute, we get

$$\begin{aligned}
\sum_{n=1}^{50}(n^2+1)\cdot n! &= (1-1)\cdot 1! + \sum_{n=1}^{50}(n^2+1)\cdot n! \\
&= (2-1)\cdot 2! + \sum_{n=2}^{50}(n^2+1)\cdot n! \\
&= (3-1)\cdot 3! + \sum_{n=3}^{50}(n^2+1)\cdot n! \\
&\vdots \\
&= (50-1)\cdot 50! + \sum_{n=50}^{50}(n^2+1)\cdot n! \\
&= (51-1)\cdot 51!
\end{aligned}$$

The final answer is then $\frac{50\cdot 51!}{50!} = 50\cdot 51 = 2550$

27. **[19]** Call a number alternating if each digit is either greater than or less than all of its adjacent digits and no two digits are equal. For example, 19283 and 91827 are alternating. Compute the largest alternating multiple of 11.

Solution: First, let's think recall the divisibility rule of 11: The sum of the digits in the odd places and the sum of the digits in the even places differ by a multiple of 11. Let n equal the largest alternating multiple of 11. Let x equal the sum of the even digits of n. In order for n to be maximal, it should have 10 digits. So, the total sum of the digits would be $0+1+2+...+9=45$. So, $x-(45-x)$ must be a multiple of 11. So, $2x \equiv 45 \bmod 11$, and $x \equiv 6 \bmod 11$. Since x is the sum of five distinct non-negative integers at most nine, x must be between 10 and 35. Combining our conditions, x is either 17 or 28.

We can then consider the first digit of our number.

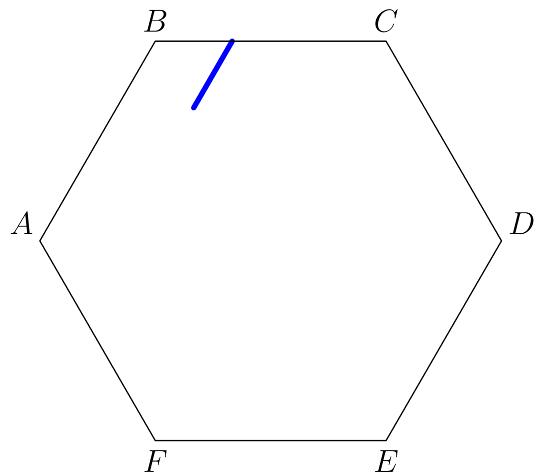
Claim the first: The first digit cannot be 9.
If the first digit is 9, the second digit must be smaller. So, the odd place digits are all larger than their neighboring even place digits. So, 8 must be in an odd place, as there is only one number larger than it, and 8 is not in the first slot. Then, the sum of the odd places must be 28, as it is already 17. The last 3 odd place numbers must then sum to 11. Since $3+4+5 > 11$, there must be an odd digit at most 2. There also cannot be an odd place digit less than 2, as each odd place digit must have two smaller neighbors. So, 2 must be an odd place digit. Now, 3 cannot be a odd-place digit, as there are not two smaller digits that can be placed on either side of it (the 1 and 0 must go around the 2). So, the odd place digits must be 9, 8, 5, 4, 2 and the even place digits must be 7, 6, 3, 1, 0. The 7 and the 6 must both go between the 9 and the 8, as all of the other odd digits are smaller, a contradiction.

So, in order to maximize our number, we would like our first digit to be 8. From the case above, we can see that the digits 9 and 8 cannot both be in the same group of digits (even or odd places). So, the even digits must be the larger digits, and the odds the smaller. Our odd place digits thus sum to 17 and the even place digits to 28. Moreover, the second digit must be 9, as it is the only digit greater than 8. Continuing along, 7 must be an even place digit. The three remaining even place digits then sum to 12. We would like the fourth digit to be 6 to maximize the number, making the remaining even place digits 5, 4, and 3, giving the remaining odd place digits as 2, 1, and 0. So, our final number is 8967251403.

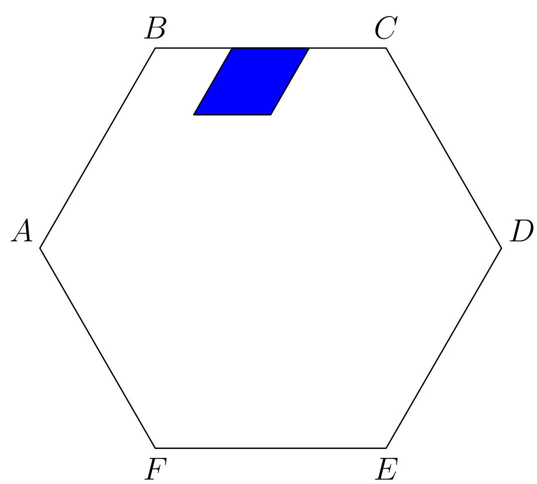
28. **[19]** Let $ABCDEF$ be a regular hexagon of side length 3. X, Y, Z are three points chosen arbitrarily on three different sides of $ABCDEF$. Compute the area of the locus of the centroid of $\triangle XYZ$.

Solution: Let's consider if X, Y, Z are on three consecutive sides (for instance, AB, BC, CD).

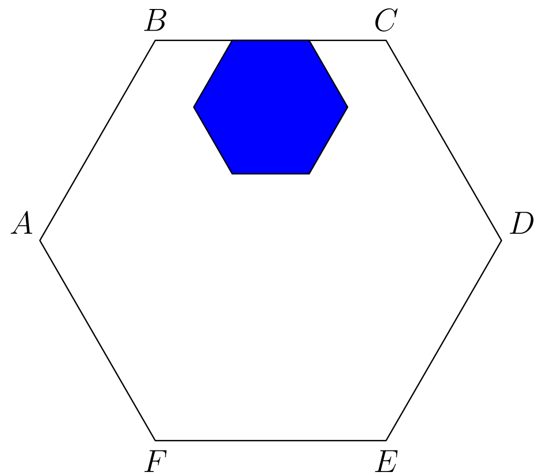
Let's fix Y and Z at B and C, respectively, and consider shifting X along segment AB from A to B. Since the centroid of $\triangle XYZ$ for $X = x,\ Y = y,\ Z = z$ can be thought of as $\frac{x+y+z}{3}$, we can notice that a shift of X along a segment of length 3 will result in a parallel shift of the centroid along a segment of length 1.



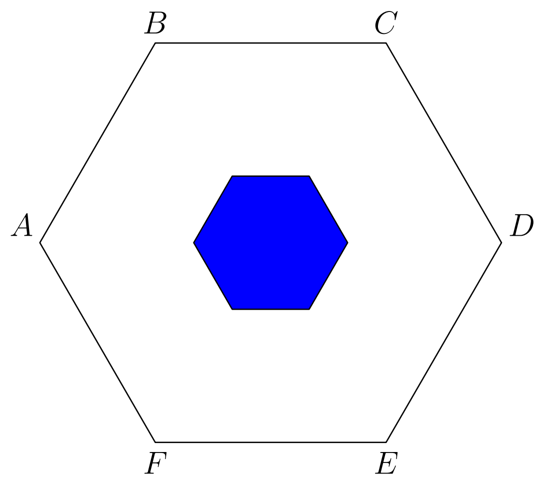
Similarly, shifting Y along BC will result in a parallel shift of length 1 along each of the points along our previous line segment, resulting in a rhombus with side length 1 and two $60°$ angles:



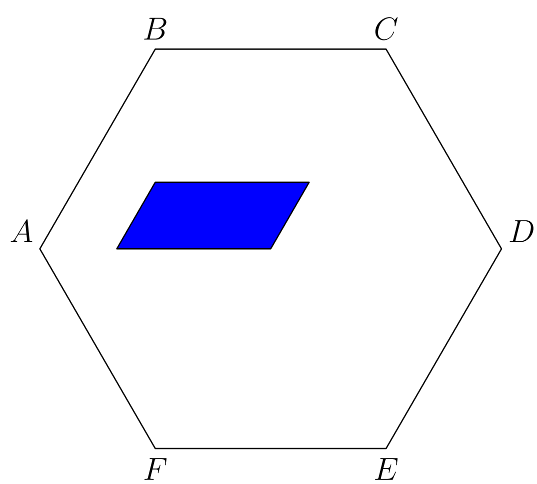
Then shifting the entire figure along a segment of length 1 parallel to CD results in a hexagon of side length 1:



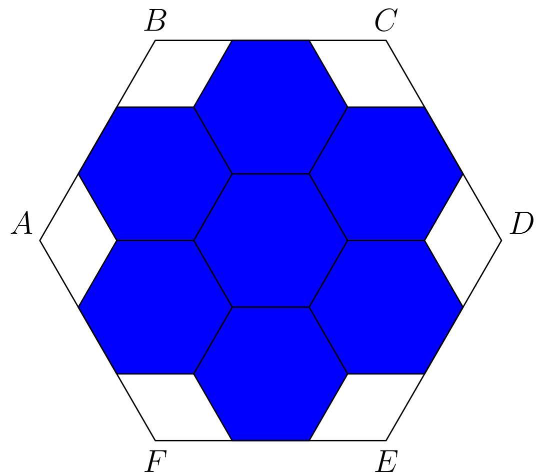
We can argue similarly that when X, Y, Z are on three non-adjacent sides (for instance, AB, CD, EF), the locus of the centroid is a hexagon of side length 1 that shares the center of $ABCDEF$:



And in the case of having two points along opposite sides (for instance, $X \in BC$, $Y \in EF$, $Z \in AB$), we have a parallelogram with side lengths 1 and 2, and two 60° angles:



Rotating for each possible set of sides, we get the following region:



Note that our third case is entirely contained within the first two. Our area is then equal to the area of seven regular hexagons of side length 1, which is $7 \cdot \frac{3\sqrt{3}}{2} = \frac{21\sqrt{3}}{2}$.

29. **[20]** How many positive integers n between 10 and 100, inclusive, satisfy

$$n \mid (n-1)!\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)?$$

Solution: If n is odd, we have

$$(n-1)!\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}\right)=$$

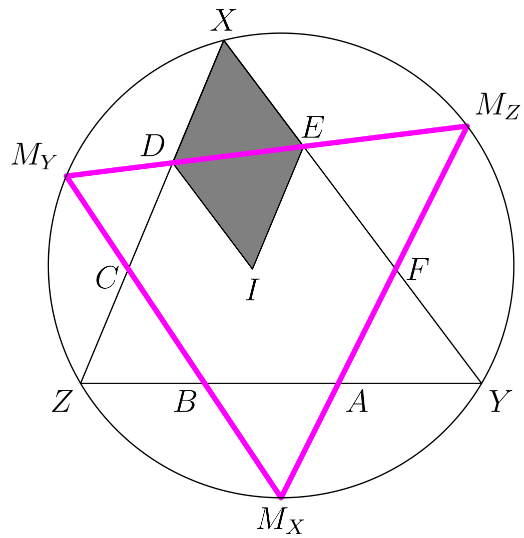
$$(n-1)!\left(\left(1+\frac{1}{n-1}\right)+\left(\frac{1}{2}+\frac{1}{n-2}\right)+\ldots+\left(\frac{1}{(n-1)/2}+\frac{1}{(n+1)/2}\right)\right)=$$

$$(n-1)!\left(\frac{n}{1(n-1)}+\frac{n}{2(n-2)}+\ldots+\frac{n}{(n-1)/2\cdot(n+1)/2}\right),$$

which is divisible by n because all terms in the sum are divisible by n. Thus, all odd numbers 45 total between 10 and 100 satisfy the condition. In the case where n is even, we can do a similar pairing of terms whose sum is divisible by n, but we are left with the term $(n-1)!\frac{1}{n/2}$. Let $k = n/2$. Then the condition becomes $2k|\frac{(2k-1)!}{k}$, or $2k^2|(2k-1)!$. For $k \geq 5$, it is clear that $(2k-1)!$ has a higher power of 2 than $2k^2$, so we only need to see when $k^2|(2k-1)!$. Let p be any prime dividing k, and let l be the largest power of p dividing k. The largest power of p dividing k^2 is $2l$, while the largest power of p dividing $(2k-1)!$ is at least $\lfloor\frac{2k-1}{p}\rfloor$. If $l > 1$, then $\lfloor\frac{2k-1}{p}\rfloor \geq \frac{k}{p} > p^l \geq 2l$. If $l = 1$, but $k \neq p$, then $\lfloor\frac{2k-1}{p}\rfloor \geq \frac{k}{p} \geq 2 = 2l$. Thus, if k isn't prime, $n = 2k$ satisfies the condition. If k is a prime, it is clear that $2k^2$ doesn't divide $(2k-1)!$. Thus, the only numbers between 10 and 100 that don't satisfy the conditions are those of the form $2p$, where p is prime. There are 13 numbers of this form: $10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94$. This gives $91 - 13 = 78$ numbers between 10 and 100 that satisfy the condition.

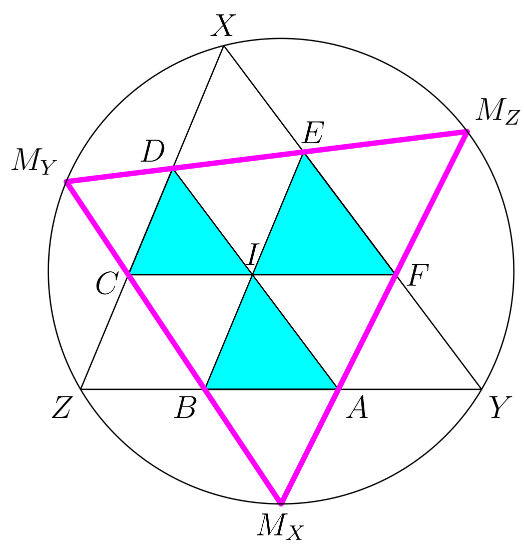
30. **[20]** Consider triangle XYZ with side lengths $13, 14, 15$. Let M_X, M_Y, M_Z be the midpoints of arcs YZ, ZX and XY in the circumcircle of XYZ. Compute the area of the hexagon formed by intersecting triangles XYZ and $M_XM_YM_Z$.

Solution: Let I be the incenter of XYZ, and let the hexagon in question be $ABCDEF$.



This problem can be split into 3 main claims.

Claim The First: $XDIE$, $YFIA$, and $ZBIC$ are all rhombii.
Note that M_YM_Z is the perpendicular bisector of XI (since $M_ZX = M_ZI$ and $M_YX = M_YI$), and that XI is an angle bisector of angle $\angle DXE$. Thus $XDIE$ is a rhombus.
Symmetric arguments prove the remaining parts of the claim.



Claim The Second: AD, BE, and CF concur at I, and $AD \parallel XY$, $BE \parallel ZX$, $CF \parallel YZ$.
Using Claim The First, $AI \parallel XY \parallel ID$, so the line AD passes through I, and $AD \parallel XY$.
Symmetric arguements prove the remaining parts of the claim.

Claim The Third: $\triangle IAB \sim \triangle ICD \sim \triangle IEF \sim \triangle XYZ$.

This follows directly from Claim The Second.

Now we will compute the answer. Note that the ratio of similarity between $\triangle IAB$ and $\triangle XYZ$ is $\frac{r}{h_x}$, where r is the inradius and h_X is the height from X to YZ.
This ratio is equivalent to $\frac{\frac{[XYZ]}{s}}{\frac{2[XYZ]}{YZ}} = \frac{YZ}{XY+YZ+ZX}$, where s is the semiperimeter of $\triangle XYZ$.
We can compute the ratios of similarity for triangles $\triangle ICD$ and $\triangle IEF$ similarly.

Now we have that the combined areas of $\triangle IAB, \triangle ICD, \triangle IEF$ is equal to $[XYZ]\frac{XY^2+YZ^2+ZX^2}{(XY+YZ+ZX)^2}$.

The remaining areas left to compute of hexagon $ABCDEF$ are the areas of triangles $\triangle IBC, \triangle IDE, \triangle IFA$. These areas are half the areas of rhombii $XDIE, YFIA, ZBIC$.

We can now conclude that the area of hexagon $ABCDEF$ is

$$\frac{[XDIE]+[YFIA]+[ZBIC]}{2}+[IAB]+[ICD]+[IEF]$$

or

$$\frac{[XDIE]+[YFIA]+[ZBIC]+2[IAB]+2[ICD]+2[IEF]}{2}$$

which is equivalent to$\frac{1}{2}([XYZ]+[IAB]+[ICD]+[IEF])$

We can simplify this to $\frac{1}{2}[XYZ](1+\frac{XY^2+YZ^2+ZX^2}{(XY+YZ+ZX)^2}) = \frac{1}{2}(84)(1+\frac{13^2+14^2+15^2}{(13+14+15)^2} = \frac{1177}{21}$