New York City Team Contest: Problems

Winter 2022

1. **[6]** Let r and s be the zeroes of $f(x) = x^2 - 7x + 10$. Compute $|\frac{r+s}{r-s}|$.

Team Name: ____________ Answer: ____________

2. **[6]** Let $ABCD$ be a square with side length 1. Let E be the midpoint of side BC, and let F be the midpoint of segment AE. Compute the area of $\triangle ADF$.

Team Name: ____________ Answer: ____________

3. **[7]** Compute

$$(123 + 231 - 321)\left(1^{(2^3)} + 2^{(3^1)} - 3^{(2^1)}\right)\left(\frac{12}{3} + \frac{23}{1} - \frac{32}{1}\right)$$

Team Name: ____________ Answer: ____________

4. **[7]** Compute the least number of line segments needed to dissect a square into 8 regions, all of which are triangles.

Team Name: ____________ Answer: ____________

5. **[8]** Julius Randle is given the ball during the waning seconds of a New York Knicks basketball game. The probability that he scores and wins the game is $\frac{1}{10}$. The probability that he turns the ball over and loses the game is $\frac{1}{10}$. The probability that he misses the shot but doesn't lose the game is $\frac{4}{5}$, repeating this in overtime. This continues until either Julius hits the shot, or loses the game. What is the probability that Julius hits the shot and wins the game for the Knicks?

Team Name: ____________ Answer: ____________

6. **[8]** Let $s(n)$ be the sum of the digits of n. Compute the sum of the 2 smallest positive integers n such that $n = 6 \cdot s(n) - 1$.

Team Name: ____________ Answer: ____________

7. **[9]** Jerry colors some set of unit lattice squares (squares of side length 1 with vertices on lattice points) purple such that any two lattice squares can be connected by a path of purple squares that touch along an edge. Given that the smallest rectangle that can be constructed with sides parallel to the axes containing all of the purple squares has area 120, find the smallest number of purple squares.

Team Name: ____________ Answer: ____________

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8. **[9]** Compute the number of ways to fill 8 crates numbered 1 though 8 with either purple pandas, teal terrapins, or grey grapes, such that each crate has exactly one object, and any crate filled with grapes is labeled with an odd number.

Team Name: ____________ Answer: ____________

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9. **[10]** John is mashing buttons on his calculator. His first button is an integer between 1 and 9; his second button is one of the symbols $+, -, \times, \div$; his third is an integer between 1 and 9. What is the probability that his button-mashed expression results in a positive integer?

Team Name: ____________ Answer: ____________

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10. **[Up to 10]** Submit two points $A : (x_1, y_1), B : (x_2, y_2)$ such that $0 \leq x_1, x_2, y_1, y_2 \leq 5\sqrt{2}$. Let L be the length of your line segment $\overline{AB}$, and let N be the number of intersections between your line segment and the segments of other teams. If your segment contains or is entirely contained within another team's segment, you will receive 0 points. Otherwise, you will receive $\dfrac{L}{N+1}$ points.

Team Name: ____________ Answer: ____________

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11. **[11]** Let $\triangle ABC$ be a triangle with $BC = 5$ and $h_B = h_C = 4$, where h_B and h_C represent the heights from B to side AC and C to side AB, respectively. Compute the area of $\triangle ABC$.

Team Name: ____________ Answer: ____________

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12. **[11]** Rishbah has a set of 13 cards number 1 through 13. He randomly choose 7 cards. Find the probability that there are 5 consecutive numbers among the chosen cards.

Team Name: ____________ Answer: ____________

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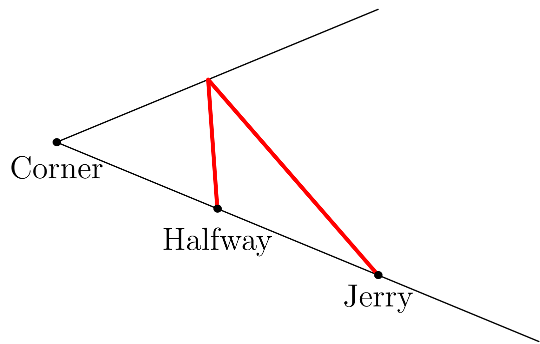
13. **[12]** Jerry is playing a game of pool on a very strange table. The table is an isosceles triangle; the two legs form a $30°$ angle. Jerry hits a ball that is on one of the legs of the triangle. The ball bounces off of the other leg before hitting the first wall halfway between its original position and the vertex of the triangle. Compute the acute angle formed between the wall and the path of the ball at the starting point. (Note that the acute angle formed between the ball's path and the wall is the same before and after colliding with the wall.)



Team Name: ____________ Answer: ____________

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14. **[12]** A figure is constructed recursively. At time $t = 0$, an equilateral triangle of side length 1 is drawn. Each minute after, an equilateral triangle of side length 1 is constructed on each side of an already existing equilateral triangle of side length 1. For example, at time $t = 2$ minutes, there are 10 equilateral triangles of side length 1 drawn. Compute the number of equilateral triangles of side length 1 drawn after 100 minutes.

Team Name: ____________ Answer: ____________

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15. **[13]** Let $\triangle ABC$ be a triangle with $AB = 3$, $BC = 4$, and $\angle ABC = 90°$. Let P be a point on line BC, and let Q be the projection of P onto line AC. If Q lies between A and C, and $\angle APQ = \frac{1}{2}\angle ACB$, compute AQ.

Team Name: ____________ Answer: ____________

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16. **[13]** Find the sum of all positive integers n such that $n + 4$ divides $4n^2 + 24n + 46$.

Team Name: ____________ Answer: ____________

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17. **[14]** Let $P(x) = x^4 - 8x^3 + 21x^2 - 14x + 6$ have roots r, s, t, u. Compute $(2 + r^2)(2 + s^2)(2 + t^2)(2 + u^2)$.

Team Name: ____________ Answer: ____________

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18. **[14]** Compute $(14!)^2 \bmod 31$.

Team Name: ____________ Answer: ____________

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19. **[15]** How many lattice paths from $(0,0)$ to $(6,6)$ do not touch the lines $y = x - \pi$ or $y = x + \pi$?

Team Name: _____________ Answer: _____________

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20. **[Up to 28]** Welcome to **USAYNO**!

Instructions: Submit a string of 6 letters corresponding to each statement: put T if you think the statement is true, F if you think it is false, and X if you do not wish to answer. You will receive $\frac{(n+1)(n+2)}{2}$ *points for n correct answers, but you will receive zero points if any of the questions you choose to answer are incorrect. Note that this means if you submit "XXXXXX" you will get one point.*

(1) For all functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, if there exists integers m and n such that $f^m(x) = x$ and $g^n(x) = x$, then there exists an integer k such that $(f \circ g)^k(x) = x$.

(2) The sum of any finite arithmetic sequence of positive integers with at least three terms is composite.

(3) There are multiple functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + 2y) = f(x) + f(y)$.

(4) Every triangle with inradius 1 and integer side lengths also has a right angle.

(5) Let $f(a)$ be a function that returns the sum of all integers b between 1 and a inclusive such that the gcd of a and b is 1. Then there exist positive integers m and n such that $f(m)f(n) = f(mn)$.

(6) For all positive integers n, and all odd prime factors p of $n^8 + 1$, we have $16|p - 1$.

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21. **[16]** A single strip of paper has the numbers $1, 2, \ldots, 10$ written on it in that order. Every minute, Max makes one cut between some two integers on a strip of paper that contains at least two of the ten integers. He then moves any piece with exactly one integer cut during that minute to the leftmost side, maintaining the order they followed after cutting. He stops once every piece has one integer on it. Find the total number of ways Max can cut the strips so that 8 is the first integer from the left when he stops.

Team Name: _____________ Answer: _____________

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22. **[16]** Let $S = a_1, a_2, a_3, \ldots$ be an infinite increasing sequence of positive integers. We have that any element a_i of the sequence is divisible by either 1071 or 1072, but it is not divisible by 107. Let k_S be the maximum possible value of $a_{i+1} - a_i$ among all i. Compute the smallest possible value of k_S, over all possible S.

Team Name: _____________ Answer: _____________

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23. **[17]** Let $\triangle ABC$ be a triangle with $AB = 4, BC = 6, CA = 5$. Let D be the intersection of the B-angle bisector and the line through A parallel to BC. Let E be the intersection of the C-angle bisector and the line through B parallel to AC. Let F be the intersection of the A-angle bisector and the line through C parallel to BA. Compute the area of (non convex) hexagon $AEBFCD$.

Team Name: _____________ Answer: _____________

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24. **[17]** Mr. Cocoros and Rishabh are taking a tour of the fourth floor, which has 8 rooms, including rooms 403 and 407. In order to avoid being supsicious, Mr. Cocoros won't take Rishabh to the same room twice on the tour.
How many ways are there for Mr. Cocoros to take Rishabh on a tour of the fourth floor that starts in room 403 ends in room 407 without being suspicious?

Team Name: _____________ Answer: _____________

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25. **[18]** Call a function $f : \mathbb{Z} \to \{0, 1, 2, 3, 4\}$ *1-multiplicative* if for every a,

- $f(a + 5) = f(a)$
- There exists at least one value $b \not\equiv a \mod 5$ for which $f(ab) \equiv f(a) \cdot f(b) \mod 5$.

Find the number of 1-multiplicative functions.

Team Name: ____________ Answer: ____________

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26. **[18]** Compute

$$\frac{\sum_{n=1}^{50} (n^2 + 1) \cdot n!}{50!}$$

Team Name: ____________ Answer: ____________

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27. **[19]** Call a number *alternating* if each digit is either greater than or less than all of its adjacent digits and no two digits are equal. For example, 19283 and 91827 are alternating. Compute the largest alternating multiple of 11.

Team Name: ____________ Answer: ____________

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28. **[19]** Let $ABCDEF$ be a regular hexagon of side length 3. X, Y, Z are three points chosen arbitrarily on three different sides of $ABCDEF$. Compute the area of the locus of the centroid of $\triangle XYZ$.

Team Name: ____________ Answer: ____________

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29. **[20]** How many positive integers n between 10 and 100, inclusive, satisfy

$$n \mid (n-1)! \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1}\right)?$$

Team Name: ____________ Answer: ____________

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30. **[20]** Consider triangle XYZ with side lengths $13, 14, 15$. Let M_X, M_Y, M_Z be the midpoints of arcs YZ, ZX and XY in the circumcircle of XYZ. Compute the area of the hexagon formed by intersecting triangles XYZ and $M_X M_Y M_Z$.

Team Name: ____________ Answer: ____________

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