Some Round Part 2!

1 Problems

1. At a math competition, a team of 8 students has 2 hours to solve 30 problems. If each problem needs to be solved by 2 students, on average how many miutes can a student spend on a problem?

2. Determine all real values of x for which

$$\frac{1}{\sqrt{x}+\sqrt{x-2}}+\frac{1}{\sqrt{x}+\sqrt{x+2}}=\frac{1}{4}$$

3. What is the largest integer with distinct digits such that no two of its digits sum to a perfect square?

4. Triangle ABC has $AB=8$, $AC=12$, $BC=10$. Let D be the intersection of the angle bisector of angle A with BC. Let M be the midpoint of BC. The line parallel to AC passing through M intersects AB at N. THe line parallel to AB passing through D intersects AC at P. MN and DP intersect at E. Find the area of $ANEP$.

5. How many decreasing sequences $a_1, a_2, \ldots, a_2019$ of positive integers are there such that $a_1 \leq 2019^2$ and $a_n + n$ is even for each $1 \leq n \leq 2019$?

6. Let ABC be a triangle with hypotenuse AB. Point E is on AB with $AE = 10B$, and point D is outside triangle ABC such that $DC = DB$ and $\angle CDA = \angle BDE$. Let $[ABC]$ and $[BCD]$ denote the areas of triangles ABC and BCD. Determine the value of $\frac{[BCD]}{[ABC]}$.

7. The infinite sequence $a_0, a_1, \ldots$ is given by $a_1 = \frac{1}{2}$, $a_{n+1} = \sqrt{\frac{1+a_n}{2}}$. Determine the infinite product $a_1 a_2 a_3 \ldots$.

8. Let $P(x)$ be a polynomial wiht integer coefficients such that

$$P(\sqrt{2}\sin x) = -P(\sqrt{2}\cos x)$$

for all real numbers x. What is the largest prime that must divide $P(2019)$?

9. Tommy takes a 25-question true-false test. He answers each question correctly with independent probability $\frac{1}{2}$. Tommy earns bonus points for correct streaks: the first question in a streak is worth 1 point, the second question is worth 2 points, and so on. For instance, the sequence TFFTTTFT is worth $1+1+2+3+1=8$ points. Compute the expected value of Tommy's score.

10. Let n be the largest positive integer such that 5^n divides $12^{2014}+13^{2014}$. Compute the remainder when $\frac{12^{2015}+13^{2015}}{5^n}$ is divided by 1000.