Some Round!

1 Problems

1. A trifecta is an ordered triple of positive integers (a, b, c) with $a < b < c$ such that a divides b, b divides c, and c divides ab. What is the largest possible sum $a + b + c$ over all trifectas of three-digit integers?

2. How many six-letter words formed from the letters of AMC do not contain the substring AMC? (For example, AMAMMC has this property, but AAMCCC does not.)

3. Seven two-digit integers form a strictly increasing arithmetic sequence. If the first and last terms of this sequence have the same set of digits, what is the sum of all possible medians of the sequence?

4. The Fibonacci sequence $F_0, F_1, \ldots$ satisfies $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Compute the number of triples (a, b, c) with $0 \leq a < b < c \leq 100$ for which F_a, F_b, F_c is an arithmetic progression.

5. Let a, b be positive real numbers with $a > b$. Compute the minimum possible value of the expression

$$\frac{a^2b - ab^2 + 8}{ab - b^2}$$

6. Determine the number of 10-letter strings consisting of As, Bs, and Cs such that there is no B between any two As.

7. In a circle of radius 10, three congruent chords bound an equilateral triangle with side length 8. The endpoints of these chords form a convex hexagon. Compute the area of this hexagon.

8. What is the product of all factors of 30^{12} that are congruent to 1 modulo 7?

9. Two circles with radii 3 and 4 are externally tangent at P. Let $A \neq P$ be on the first circle and $B \neq P$ be on the second circle and let the tangents at A and B to the respective circles intersect at Q. Given that $QA = QB$ and AB bisects PQ, compute the area of QAB.

10. Kelvin the Frog lives in the 2-D plane. Each day, he picks a uniformly random direction (i.e. a uniform random bearing $\theta \in [0, 2\pi]$) and jumps a mile in that direction. Let D be the number of miles Kelving is away from is starting point after ten days. Determine the expected value of D^4.