NYCMT 2022-2023 HW#5

NYCMT

October 28 - November 4, 2022

These problems are due Friday, November 4th. Please solve as many problems as you can, and write up solutions (**not just answers!**) to the ones you solve. Also write down any progress you make on problems you don't solve. Please write solutions for different questions on separate pages. Make sure to write your name on each page and page numbers per problem.

If you're not going to be present on November 4th, you can scan your solutions and email them to jothman30@stuy.edu and jmoltz30@stuy.edu. If you will be there, just hand in your responses on paper. If you have any questions, just ask one of us on Discord or using one of the above emails.

Problems are not necessarily in difficulty order. Make sure to try them all!

Enjoy!

1 Problems

Problem 1. Let S be a subset of the integers $1, 2, \cdots, 100$ such that there do not exist elements $a, b \in S$ such that $a = 3b$. Find, with proof, the maximum value of $|S|$ (the size of S).

Problem 2. For reals $x \geq 3$, let $f(x)$ denote the function

$$f(x) = \frac{-x + x\sqrt{4x-3}}{2}$$

Let $a_1, a_2, \ldots,$ be the sequence satisfying $a_1 > 3$, $a_{2023} = 12345$, and for $n = 1, 2, \ldots, 2022$, $a_{n+1} = f(a_n)$. Determine the value of

$$a_1 + \sum_{i=1}^{2022} \frac{a_{i+1}^3}{a_i^2 + a_i a_{i+1} + a_{i+1}^2}$$

Problem 3. Let S be the set of lattice points P (points with integer coordinates) of the form $P = (a, b, c)$ satisfying $0 \leq a \leq 2$, $0 \leq b \leq 3$, and $0 \leq c \leq 4$. If P_1 and P_2 are two distinct points of S chosen uniformly and at random, compute the probability that the midpoint of P_1P_2 is in S as well.

Problem 4. Josiah's drawer contains at most 2022 socks, some of which are red and the rest of which are blue. Josiah draws two socks from the drawer without replacement, uniformly and at random. If the probability that Josiah's socks are the same color is $\frac{1}{2}$, compute the maximum possible value of the number of socks in his drawer.

Problem 5. Let $ABCD$ be a rectangle. Circles with diameters AB and CD meet at points P and Q inside the rectangle such that P is closer to segment BC than Q. Let M and N be the midpoints of segments AB and CD. If $\angle MPN = 40^\circ$, find the degree measure of $\angle BPC$.